

QUATERNIONIC SOLITON EQUATIONS FROM HAMILTONIAN CURVE FLOWS IN \mathbb{HP}^n

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ABSTRACT. A bi-Hamiltonian hierarchy of quaternion soliton equations is derived from geometric non-stretching flows of curves in the quaternionic projective space \mathbb{HP}^n . The derivation adapts the method and results in recent work by one of us on the Hamiltonian structure of non-stretching curve flows in Riemannian symmetric spaces $M = G/H$ by viewing $\mathbb{HP}^n \simeq \mathrm{U}(n+1, \mathbb{H})/\mathrm{U}(1, \mathbb{H}) \times \mathrm{U}(n, \mathbb{H}) \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ as a symmetric space in terms of compact real symplectic groups and quaternion unitary groups. As main results, scalar-vector (multi-component) versions of the sine-Gordon (SG) equation and the modified Korteweg-de Vries (mKdV) equation are obtained along with their bi-Hamiltonian integrability structure consisting of a shared hierarchy of quaternionic symmetries and conservation laws generated by a hereditary recursion operator. The corresponding geometric curve flows in \mathbb{HP}^n are shown to be described by a non-stretching wave map and a mKdV analog of a non-stretching Schrödinger map.

1. INTRODUCTION AND SUMMARY

Certain geometric flows of curves in homogeneous plane and space geometries are well-known to encode scalar soliton equations through the induced evolution of geometrical invariants of the curve [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. There has been much recent interest in extending such geometrical derivations to multi-component soliton equations. For example, the two vector versions of the modified Korteweg-de Vries (mKdV) equation known from symmetry-integrability classifications [11] have been derived [12] from non-stretching curve flows in the homogeneous Riemannian geometries $M = \mathrm{SO}(N+1)/\mathrm{SO}(N) \simeq S^N$ and $M = \mathrm{SU}(N)/\mathrm{SO}(N)$, generalizing a similar derivation of the scalar mKdV equation obtained previously [13] in the case of the standard 2-sphere geometry, $S^2 \simeq \mathrm{SO}(3)/\mathrm{SO}(2) \simeq \mathrm{SU}(2)/\mathrm{SO}(2)$. The same approach has also been used to provide a geometric origin for the scalar sine-Gordon (SG) equation and its two known vector versions [14, 12]. Related work on integrable curve flows in Euclidean and spherical Riemannian geometries, \mathbb{R}^N and S^N , has appeared in [15, 16, 17, 18, 19].

The derivation of vector mKdV and SG equations in the geometries $M = G/\mathrm{SO}(N)$ is based on a moving parallel frame formulation for arclength-parameterized curves γ which is closely analogous to the standard parallel framing of curves in Euclidean geometry [20, 15, 17]. In this generalization [12], the Euclidean group is replaced by the respective isometry groups $G = \mathrm{SO}(N+1)$ and $G = \mathrm{SU}(N)$ whose rotation subgroups $\mathrm{SO}(N) \subset G$ act as the gauge group for the frame bundle of $M = G/\mathrm{SO}(N)$ (analogously to that of Euclidean space). Given a parallel framing, where x is the G -invariant arclength along γ and $T = \gamma_x$

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is the unit tangent vector, the Cartan structure equations for torsion and curvature of the Riemannian connections ∇_x, ∇_t on the 2-dimensional surface of any non-stretching curve flow $\gamma(t, x)$ in the manifold $M = G/SO(N)$ are seen [12] to geometrically encode a pair of compatible Hamiltonian operators. This bi-Hamiltonian structure generates a hierarchy of integrable curve flows in which the frame components of the principal normal vector $N = \nabla_x T$ satisfy vector evolution equations related by a hereditary recursion operator. Each evolution equation displays invariance under the rotation group $SO(N-1)$ (in the $SO(N)$ gauge group) that acts in the normal space at each point on the curve γ and preserves the form of the connection matrix of the parallel frame, so thus the principal normal components have the meaning of geometrical covariants that are determined by γ up to $SO(N-1)$ gauge freedom (which represents the equivalence group of the framing). The lowest-order flow in the respective hierarchies in $M = SO(N+1)/SO(N) \simeq S^N$ and $M = SU(N)/SO(N)$ yields two different vector mKdV equations which are $SO(N-1)$ -invariant. In addition, each hierarchy also contains a flow that yields a $SO(N-1)$ -invariant vector SG equation arising from the kernel of the hereditary recursion operator. The corresponding geometric curve flows $\gamma(t, x)$ in both hierarchies are found to be given by [13, 12] wave maps and mKdV analogs of Schrödinger maps on $M = G/SO(N)$.

Complex versions of these vector SG equations and vector mKdV equations have been derived [21] through applying the same method to geometric non-stretching curve flows in the Lie groups $G = SO(N+1), SU(N)$ themselves viewed as homogeneous Riemannian geometries in the standard manner $G \simeq G \times G / \text{diag}(G \times G)$. This approach has also yielded the two known vector versions of the nonlinear Schrödinger equation (NLS), along with their bi-Hamiltonian integrability structure.

A broad generalization of such results has been obtained in recent work [22] by one of us on the Hamiltonian structure of non-stretching flows of curves in homogeneous Riemannian geometries (i.e. symmetric spaces) $M = G/H$, including compact semisimple Lie group geometries $M = K$ for $G = K \times K, H = \text{diag } G$. For all these geometries, the results give a general geometric derivation of group-invariant (multi-component) SG, mKdV, and NLS equations along with their bi-Hamiltonian integrability structure consisting of a shared hierarchy of symmetries and conservation laws generated by a group-invariant recursion operator.

In the present paper we adapt this derivation to non-stretching curve flows in the quaternion projective geometry $M = \mathbb{HP}^n$ by means of the symmetric space isomorphisms

$$\mathbb{HP}^n \simeq \text{U}(n+1, \mathbb{H}) / \text{U}(1, \mathbb{H}) \times \text{U}(n, \mathbb{H}) \simeq \text{Sp}(n+1) / \text{Sp}(1) \times \text{Sp}(n) \quad (1.1)$$

given in terms of Hamilton's quaternions $\mathbb{H} = \text{span}(1, i, j, k)$. Underlying these isomorphisms is a basic identification [23] between the quaternion unitary Lie algebra $\mathfrak{u}(n, \mathbb{H})$ and the compact symplectic real Lie algebra $\mathfrak{sp}(n)$ holding for $n \geq 1$. Our main result will be to obtain bi-Hamiltonian geometric curve flows in $M = \mathbb{HP}^n$ yielding an integrable multi-component quaternionic version of the mKdV equation and the SG equation.

A summary of relevant properties of symplectic groups, quaternion unitary groups, and the symmetric space structure of \mathbb{HP}^n is provided in section 2. In section 3, starting from the gauge group $\text{U}(1, \mathbb{H}) \times \text{U}(n, \mathbb{H})$ of the frame bundle of \mathbb{HP}^n , we extend the moving parallel frame formulation to curves in \mathbb{HP}^n and explain its main geometrical properties. In particular, the components of the principal normal vector along any arclength-parameterized

curve with this framing define geometrical covariants that transform as a quaternion scalar-vector pair (or a single quaternion scalar in the case $n = 1$) with respect to the equivalence group $U(1, \mathbb{H}) \times U(n - 1, \mathbb{H})$ (which preserves the form of the parallel connection matrix) of the framing. In section 4 we next show how the Cartan structure equations for framed curve flows encode two compatible quaternion Hamiltonian operators. We then use these operators in section 5 to generate a hierarchy of quaternion Hamiltonian vector fields and corresponding quaternion scalar-vector evolution equations with an explicit bi-Hamiltonian structure. This yields a quaternion scalar-vector mKdV equation and a quaternion scalar-vector SG equation, both of which are unitarily invariant under $U(1, \mathbb{H}) \times U(n - 1, \mathbb{H})$ and share a hierarchy of quaternionic symmetries and conservation laws which are generated by a $U(1, \mathbb{H}) \times U(n - 1, \mathbb{H})$ -invariant recursion operator. In section 6 we derive the corresponding geometric SG and mKdV curve flows in $\mathbb{H}\mathbb{P}^n$ and show that these flows respectively are described by a non-stretching wave map and a non-stretching mKdV map. Finally, we make some concluding remarks in section 7.

2. ALGEBRAIC AND GEOMETRIC PRELIMINARIES

Here we collect some useful facts about the relevant algebraic and geometric structure of $\mathbb{H}\mathbb{P}^n$ as a quaternion symmetric space.

The complex symplectic group $\mathrm{Sp}(n, \mathbb{C})$ is the group of matrices g in $\mathrm{GL}(2n, \mathbb{C})$ that leaves invariant the exterior form $z_1 \wedge z_{n+1} + \cdots + z_n \wedge z_{2n}$ in terms of coordinates $(z_1, \dots, z_{2n}) \in \mathbb{C}^{2n}$, i.e.

$$g^t J_n g = J_n, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Recall that the group of matrices g in $\mathrm{GL}(2n, \mathbb{C})$ that leaves invariant the Hermitian form $z_1 \bar{z}_1 + \cdots + z_{2n} \bar{z}_{2n}$ is the complex unitary group $U(2n)$, i.e.

$$g^t \bar{g} = I_{2n}.$$

The compact *symplectic group* is defined by $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap U(2n)$, whose elements have the form

$$g = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \quad A^t \bar{A} + \bar{B}^t B = I_n, \quad A^t \bar{B} - \bar{B}^t A = 0. \quad (2.1)$$

In the case $n = 1$, the compact symplectic group $\mathrm{Sp}(1)$ has a well-known identification with Hamilton's quaternions. The quaternions are defined as an associative non-commutative algebra \mathbb{H} of dimension 4 over the real numbers, with generators i, j, k that satisfy the multiplicative relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Quaternions $q \in \mathbb{H} = \mathrm{span}(1, i, j, k)$ have two main properties: First, every q has a natural complex conjugate defined by the conjugation relations

$$\bar{i} = -i, \quad \bar{j} = -j, \quad \bar{k} = -k, \quad \bar{1} = 1,$$

whereby

$$\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1.$$

Second, each non-zero q has a multiplicative inverse given by $q^{-1} = \bar{q}/|q|^2$, where $|q| = (q\bar{q})^{1/2}$ is the norm of q . Thus, \mathbb{H} is a normed, hypercomplex, division algebra.

Proposition 2.1. *There is a group isomorphism*

$$g = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} = \begin{pmatrix} A_1 + A_2i & B_1 + B_2i \\ -B_1 + B_2i & A_1 - A_2i \end{pmatrix} \longleftrightarrow \tilde{g} = A + Bj = A_1 + A_2i + B_1j + B_2k \quad (2.2)$$

under which the compact symplectic group $\mathrm{Sp}(n)$ is identified with the quaternionic unitary group $\mathrm{U}(n, \mathbb{H})$ given by all matrices $\tilde{g} \in \mathrm{GL}(n, \mathbb{H})$ satisfying $\tilde{g}^t \tilde{g} = I_n$. In particular, $\mathrm{Sp}(1)$ is identified with the group formed by unit-norm elements in \mathbb{H} ,

$$g = \begin{pmatrix} a_1 + a_2i & b_1 + b_2i \\ -b_1 + b_2i & a_1 - a_2i \end{pmatrix} \longleftrightarrow \tilde{g} = a_1 + a_2i + b_1j + b_2k \quad (2.3)$$

where $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1 = |\tilde{g}|^2$.

The group $\mathrm{U}(n, \mathbb{H}) \simeq \mathrm{Sp}(n)$ acts on \mathbb{H}^n by right multiplication, i.e. for $\mathbf{x} \in \mathbb{H}^n$ and $\tilde{g} \in \mathrm{U}(n, \mathbb{H})$, $\mathbf{x} \mapsto \mathbf{x}\tilde{g} \in \mathbb{H}^n$. There is a natural Hermitian inner product on \mathbb{H}^n defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}\overline{\mathbf{y}}^t$$

which is invariant under the group action of $\mathrm{U}(n, \mathbb{H})$, i.e. $\langle \mathbf{x}\tilde{g}, \mathbf{y}\tilde{g} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\tilde{g} \in \mathrm{U}(n, \mathbb{H})$. This inner product is the sum of a Euclidean inner product given by

$$\mathrm{Re} \langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2} \langle \mathbf{x}, \mathbf{y} \rangle + \frac{1}{2} \langle \mathbf{y}, \mathbf{x} \rangle = \frac{1}{2} (\mathbf{x}\overline{\mathbf{y}}^t + \mathbf{y}\overline{\mathbf{x}}^t) \quad (2.4)$$

and an imaginary skew-form given by

$$\mathrm{Im} \langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2} \langle \mathbf{x}, \mathbf{y} \rangle - \frac{1}{2} \langle \mathbf{y}, \mathbf{x} \rangle = \frac{1}{2} (\mathbf{x}\overline{\mathbf{y}}^t - \mathbf{y}\overline{\mathbf{x}}^t). \quad (2.5)$$

With respect to the Euclidean inner product, there is an orthogonal decomposition of \mathbb{H}^n into real and imaginary parts $\mathbb{H}^n = \mathbb{R}^n \oplus \mathbb{Q}^n$ where

$$\mathbb{Q} := \{q \in \mathbb{H} \mid q + \overline{q} = 0\} = \mathrm{span}(i, j, k) \quad (2.6)$$

will denote the set of imaginary quaternions.

Proposition 2.2.

1. The Lie algebra $\mathfrak{g} = \mathfrak{sp}(n)$ of $G = \mathrm{Sp}(n)$ consists of all matrices $g \in \mathfrak{gl}(2n, \mathbb{C})$ satisfying

$$gJ_n + J_ng^t = 0, \quad g + \overline{g}^t = 0.$$

2. The group isomorphism $\mathrm{Sp}(n) \simeq \mathrm{U}(n, \mathbb{H})$ induces a Lie algebra isomorphism between the symplectic Lie algebra $\mathfrak{sp}(n)$ and the quaternionic unitary Lie algebra $\mathfrak{u}(n, \mathbb{H})$ consisting of all matrices $\tilde{g} \in \mathfrak{gl}(n, \mathbb{H})$ satisfying $\tilde{g} + \overline{\tilde{g}}^t = 0$. This isomorphism is explicitly given by

$$g = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} = \begin{pmatrix} A_1 + A_2i & B_1 + B_2i \\ -B_1 + B_2i & A_1 - A_2i \end{pmatrix} \longleftrightarrow \tilde{g} = A + Bj = A_1 + A_2i + B_1j + B_2k \quad (2.7)$$

where $B = B^t, A + \overline{A}^t = 0$. In particular, the Lie algebra $\mathfrak{sp}(1)$ is identified with $\mathfrak{u}(1, \mathbb{H}) = \mathbb{Q}$.

3. The Cartan-Killing form on $\mathfrak{g} = \mathfrak{sp}(n) \simeq \mathfrak{u}(n, \mathbb{H})$ is given by

$$\langle g_1, g_2 \rangle = \mathrm{tr}(\mathrm{ad}(g_1)\mathrm{ad}(g_2)) = 2(n+1)\mathrm{tr}(g_1g_2) = 4(n+1)\mathrm{Re}(\mathrm{tr}(\tilde{g}_1\tilde{g}_2)) = \langle \tilde{g}_1, \tilde{g}_2 \rangle \quad (2.8)$$

for $g_1 \leftrightarrow \tilde{g}_1, g_2 \leftrightarrow \tilde{g}_2$ given by (2.7). In particular, the negative-definite Cartan-Killing norm is explicitly given by

$$\langle g, g \rangle_{\mathfrak{sp}(n)} = -4(n+1)\mathrm{tr}(A\overline{A}^t + B\overline{B}^t) = -4(n+1)\mathrm{tr}(A_1A_1^t + A_2A_2^t + B_1B_1^t + B_2B_2^t) = \langle \tilde{g}, \tilde{g} \rangle_{\mathfrak{u}(n, \mathbb{H})}. \quad (2.9)$$

4. For the case $n = 1$, where $g = \begin{pmatrix} a_2 i & b_1 + b_2 i \\ -b_1 + b_2 i & -a_2 i \end{pmatrix} \in \mathfrak{sp}(1)$ is identified with $q = a_2 i + b_1 j + b_2 k \in \mathfrak{u}(1, \mathbb{H}) = \mathbb{Q}$, the norm is given by

$$-\frac{1}{8}\langle g, g \rangle_{\mathfrak{sp}(1)} = a_2^2 + b_1^2 + b_2^2 = |q|^2 = \langle q, q \rangle. \quad (2.10)$$

The Lie group $U(n+1, \mathbb{H})$ arises geometrically as the isometry group of the quaternionic projective space $\mathbb{H}\mathbb{P}^n$. This space consists of the points on the unit sphere in \mathbb{H}^{n+1} with the identification of pairs of points x and $x\tilde{g}$ for every group element $\tilde{g} \in U(1, \mathbb{H})$. Since the symmetry group of the Hermitian inner product on \mathbb{H}^{n+1} is $G = U(n+1, \mathbb{H})$, then the action of G modulo the action of $U(1, \mathbb{H})$ represents the non-trivial isometries of $\mathbb{H}\mathbb{P}^n$. Moreover, if we consider the origin in $\mathbb{H}\mathbb{P}^n$ as represented by the point $o = (1, 0, \dots, 0) \in \mathbb{H}^{n+1}$, then the isotropy subgroup of $U(n+1, \mathbb{H})$ leaving invariant this point $x = o$ is given by $H = U(1, \mathbb{H}) \times U(n, \mathbb{H})$. Hence $\mathbb{H}\mathbb{P}^n$ is a homogeneous quaternionic manifold

$$\mathbb{H}\mathbb{P}^n = \frac{U(n+1, \mathbb{H})}{U(1, \mathbb{H}) \times U(n, \mathbb{H})} \quad (2.11)$$

on which $H^* \simeq \text{Ad}(U(1, \mathbb{H}) \times U(n, \mathbb{H}))$ acts as the isotropy group at the origin $x = o$, i.e. H^* linearly maps the tangent space $T_o\mathbb{H}\mathbb{P}^n$ into itself. It is also clear that the group element

$$S := \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \in U(n+1, \mathbb{H}) \quad (2.12)$$

yields an involutive automorphism of $U(n+1, \mathbb{H})$. Consequently, through Proposition 2.1, $\mathbb{H}\mathbb{P}^n$ has the structure of a symmetric Riemannian space

$$\mathbb{H}\mathbb{P}^n \simeq \frac{\text{Sp}(n+1)}{\text{Sp}(1) \times \text{Sp}(n)}. \quad (2.13)$$

The quaternionic structure of this space takes the form of a triple of linear maps I, J, K on $T_x\mathbb{H}\mathbb{P}^n$ having the properties $IJ = K, JK = I, KI = J, I^2 = J^2 = K^2 = -\text{id}$, as canonically associated with the action of the isotropy subgroup $\text{Ad}(\text{Sp}(1)) \subset H^*$ on $T_o\mathbb{H}\mathbb{P}^n$.

At the Lie algebra level, there is a decomposition of $\mathfrak{g} = \mathfrak{u}(n+1, \mathbb{H}) \simeq \mathfrak{sp}(n+1)$ as a symmetric Lie algebra constructed in terms of the involutive automorphism $\sigma(g) = SgS$. The eigenvalues of σ are 1 and -1 with corresponding eigenspaces

$$\mathfrak{h} := \mathfrak{u}(1, \mathbb{H}) \oplus \mathfrak{u}(n, \mathbb{H}) \simeq \mathfrak{sp}(1) \oplus \mathfrak{sp}(n), \quad \sigma(\mathfrak{h}) = \mathfrak{h} \quad (2.14)$$

and

$$\mathfrak{m} := \mathfrak{u}(n+1, \mathbb{H}) / (\mathfrak{u}(1, \mathbb{H}) \oplus \mathfrak{u}(n, \mathbb{H})) \simeq \mathfrak{sp}(n+1) / (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)), \quad \sigma(\mathfrak{m}) = -\mathfrak{m}. \quad (2.15)$$

Therefore, with respect to the Cartan-Killing form on \mathfrak{g} , σ induces an orthogonal decomposition given by the direct sum of vector spaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with Lie bracket relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \quad (2.16)$$

Moreover, the Lie subalgebra \mathfrak{h} is identified with the generators of isometries that leave fixed the origin in $\mathbb{H}\mathbb{P}^n$ (i.e. yielding the action of $H^* : T_o\mathbb{H}\mathbb{P}^n \rightarrow T_o\mathbb{H}\mathbb{P}^n$), while the vector space \mathfrak{m} is identified with the generators of isometries that carry the origin o to any point $x \neq o$ in $\mathbb{H}\mathbb{P}^n$.

Lemma 2.3.

1. The matrix representation of the vector space \mathfrak{m} and the Lie subalgebra \mathfrak{h} of \mathfrak{g} in $\mathfrak{gl}(n+1, \mathbb{H})$ is given by

$$(\mathfrak{m}, \mathfrak{m}) := \begin{pmatrix} 0 & \mathfrak{m} & \mathfrak{m} \\ -\overline{\mathfrak{m}} & 0 & \mathbf{0} \\ -\overline{\mathfrak{m}}^t & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathfrak{m}, \quad (2.17)$$

$$(\mathfrak{p}, \mathfrak{q}, \mathbf{h}, \mathbf{H}) := \begin{pmatrix} \mathfrak{p} & 0 & \mathbf{0} \\ 0 & \mathfrak{q} & \mathbf{h} \\ \mathbf{0} & -\overline{\mathbf{h}}^t & \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathfrak{p} & 0 & \mathbf{0} \\ 0 & 0 & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \mathbf{0} \\ 0 & \mathfrak{q} & \mathbf{h} \\ \mathbf{0} & -\overline{\mathbf{h}}^t & \mathbf{H} \end{pmatrix} \in \mathfrak{h} = \mathfrak{u}(1, \mathbb{H}) \oplus \mathfrak{u}(n, \mathbb{H}), \quad (2.18)$$

in which $\mathbf{m}, \mathbf{h} \in \mathbb{H}^{n-1}$ are quaternionic vectors, $\mathfrak{m} \in \mathbb{H}$ is a quaternionic scalar, $\mathfrak{p}, \mathfrak{q} \in \mathbb{Q}$ are imaginary quaternions, and $\mathbf{H} \in \mathfrak{u}(n-1, \mathbb{H})$ is a quaternionic matrix. The bracket relations (2.16) take the form:

$$[(\mathfrak{m}, \mathfrak{m}), (\mathfrak{p}, \mathfrak{q}, \mathbf{h}, \mathbf{H})] = (\mathfrak{m}\mathfrak{q} - \mathfrak{p}\mathfrak{m} - \langle \mathbf{m}, \mathbf{h} \rangle, \mathfrak{m}\mathbf{h} - \mathfrak{p}\mathbf{m} + \mathbf{m}\mathbf{H}) \in \mathfrak{m} \quad (2.19a)$$

$$[(\mathfrak{m}_1, \mathbf{m}_1), (\mathfrak{m}_2, \mathbf{m}_2)] = (-\mathfrak{m}_1\overline{\mathfrak{m}}_2 + \mathfrak{m}_2\overline{\mathfrak{m}}_1 - 2\text{Im} \langle \mathbf{m}_1, \mathbf{m}_2 \rangle, -\overline{\mathfrak{m}}_1\mathfrak{m}_2 + \overline{\mathfrak{m}}_2\mathfrak{m}_1, -\overline{\mathfrak{m}}_1\mathbf{m}_2 + \overline{\mathfrak{m}}_2\mathbf{m}_1) \in \mathfrak{h}, \quad (2.19b)$$

$$[(\mathfrak{p}_1, \mathfrak{q}_1, \mathbf{h}_1, \mathbf{H}_1), (\mathfrak{p}_2, \mathfrak{q}_2, \mathbf{h}_2, \mathbf{H}_2)] = (\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_2\mathfrak{p}_1, \mathfrak{q}_1\mathfrak{q}_2 - \mathfrak{q}_2\mathfrak{q}_1 - 2\text{Im} \langle \mathbf{h}_1, \mathbf{h}_2 \rangle, \mathfrak{q}_1\mathbf{h}_2 - \mathfrak{q}_2\mathbf{h}_1 + \mathbf{h}_1\mathbf{H}_2 - \mathbf{h}_2\mathbf{H}_1, \overline{\mathbf{h}}_1^t\mathbf{h}_2 - \overline{\mathbf{h}}_2^t\mathbf{h}_1 + \mathbf{H}_1\mathbf{H}_2 - \mathbf{H}_2\mathbf{H}_1) \in \mathfrak{h}. \quad (2.19c)$$

2. The restriction of Cartan-Killing form on \mathfrak{g} to \mathfrak{m} yields a negative-definite inner product

$$\langle (\mathfrak{m}_1, \mathbf{m}_1), (\mathfrak{m}_2, \mathbf{m}_2) \rangle = -\chi \text{Re}(\mathfrak{m}_1\overline{\mathfrak{m}}_2 + \langle \mathbf{m}_1, \mathbf{m}_2 \rangle) \quad (2.20)$$

where

$$\chi = 8(n+2). \quad (2.21)$$

3. The rank of \mathfrak{m} is equal to 1.

Cartan subspaces of \mathfrak{m} are defined as a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{m}$, having the property that it is the centralizer of its elements, $\mathfrak{a} = \mathfrak{m} \cap \mathfrak{c}(\mathfrak{a})$. It can be shown [24] that any two Cartan subspaces are isomorphic to one another under some linear transformation in $\text{Ad}(H)$ and that the action of $\text{Ad}(H)$ on any Cartan subspace \mathfrak{a} generates \mathfrak{m} . Since $\mathfrak{m} = \mathfrak{u}(n+1, \mathbb{H})/(\mathfrak{u}(1, \mathbb{H}) \oplus \mathfrak{u}(n, \mathbb{H}))$ has $\text{rank}(\mathfrak{m}) = 1$, its Cartan subspaces \mathfrak{a} are one-dimensional.

A choice of basis element defining $\mathfrak{a} = \text{span}(\mathfrak{e})$ is given by

$$\mathfrak{e} := (1, \mathbf{0}) = \begin{pmatrix} 0 & 1 & \mathbf{0} \\ -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathfrak{m}, \quad (2.22)$$

which can be readily verified to satisfy

$$\mathfrak{m} \cap \mathfrak{c}(\mathfrak{e}) = \text{span}(\mathfrak{e}), \quad \mathfrak{m} = \text{span}(\text{Ad}(U(1, \mathbb{H}) \times U(n, \mathbb{H}))\mathfrak{e}) \quad (2.23)$$

where $\text{Ad}(h)\mathfrak{e} = h\mathfrak{e}h^{-1}$ for all $h = \exp((\mathfrak{p}, \mathfrak{q}, \mathbf{h}, \mathbf{H})) \in U(1, \mathbb{H}) \times U(n, \mathbb{H})$ with $\mathfrak{p}, \mathfrak{q} \in \mathbb{Q}$, $\mathbf{h} \in \mathbb{H}^{n-1}$, $\mathbf{H} \in \mathfrak{u}(n-1, \mathbb{H})$. Following the notation used in [22], let the centralizer subspaces

of e in \mathfrak{m} and \mathfrak{h} be denoted

$$\mathfrak{m}_{\parallel} := \mathfrak{c}(e) \cap \mathfrak{m}, \quad \mathfrak{h}_{\parallel} := \mathfrak{c}(e) \cap \mathfrak{h}. \quad (2.24)$$

The orthogonal complements (perp spaces) of these subspaces \mathfrak{m}_{\parallel} and \mathfrak{h}_{\parallel} with respect to the Cartan-Killing form will be denoted by \mathfrak{m}_{\perp} and \mathfrak{h}_{\perp} , where

$$\mathfrak{m} = \mathfrak{m}_{\parallel} \oplus \mathfrak{m}_{\perp}, \quad \mathfrak{h} = \mathfrak{h}_{\parallel} \oplus \mathfrak{h}_{\perp}.$$

Their matrix representation is given by

$$(\mathfrak{m}_{\parallel}) := \begin{pmatrix} 0 & \mathfrak{m}_{\parallel} & \mathbf{0} \\ -\mathfrak{m}_{\parallel} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{pmatrix} \in \mathfrak{m}_{\parallel}, \quad (\mathfrak{m}_{\perp}, \mathbf{m}_{\perp}) := \begin{pmatrix} 0 & \mathfrak{m}_{\perp} & \mathbf{m}_{\perp} \\ \mathfrak{m}_{\perp} & 0 & \mathbf{0} \\ -\overline{\mathbf{m}}_{\perp}^t & \mathbf{0} & 0 \end{pmatrix} \in \mathfrak{m}_{\perp}, \quad (2.25)$$

and

$$(\mathfrak{h}_{\parallel}, \mathbf{H}_{\parallel}) := \begin{pmatrix} \mathfrak{h}_{\parallel} & 0 & \mathbf{0} \\ 0 & \mathfrak{h}_{\parallel} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{\parallel} \end{pmatrix} \in \mathfrak{h}_{\parallel}, \quad (\mathfrak{h}_{\perp}, \mathbf{h}_{\perp}) := \begin{pmatrix} \mathfrak{h}_{\perp} & 0 & \mathbf{0} \\ 0 & -\mathfrak{h}_{\perp} & \mathbf{h}_{\perp} \\ \mathbf{0} & -\overline{\mathbf{h}}_{\perp}^t & 0 \end{pmatrix} \in \mathfrak{h}_{\perp}, \quad (2.26)$$

where $\mathfrak{m}_{\parallel} \in \mathbb{R}$ is a real quaternion, $\mathfrak{h}_{\parallel}, \mathfrak{m}_{\perp}, \mathfrak{h}_{\perp} \in \mathbb{Q}$ are imaginary quaternions, $\mathbf{m}_{\perp}, \mathbf{h}_{\perp} \in \mathbb{H}^{n-1}$ are quaternionic vectors, and $\mathbf{H}_{\parallel} \in \mathfrak{u}(n-1, \mathbb{H})$ is a quaternionic matrix. The corresponding decomposition of $\mathfrak{m} \simeq \mathbb{H}^n$ as a vector space is given by

$$\mathfrak{m}_{\parallel} \simeq \mathbb{R}, \quad \mathfrak{m}_{\perp} \simeq \mathbb{Q} \oplus \mathbb{H}^{n-1}. \quad (2.27)$$

The linear operator $\text{ad}(e)$ maps \mathfrak{h}_{\perp} into \mathfrak{m}_{\perp} , and vice versa. In particular,

$$\text{ad}(e)(\mathfrak{h}_{\perp}, \mathbf{h}_{\perp}) = (-2\mathfrak{h}_{\perp}, \mathbf{h}_{\perp}) \in \mathfrak{m}_{\perp}, \quad \text{ad}(e)(\mathfrak{m}_{\perp}, \mathbf{m}_{\perp}) = (2\mathfrak{m}_{\perp}, -\mathbf{m}_{\perp}) \in \mathfrak{h}_{\perp}. \quad (2.28)$$

Consequently, $\text{ad}(e)^2$ is well-defined as a linear mapping of the subspaces \mathfrak{h}_{\perp} and \mathfrak{m}_{\perp} into themselves.

Let H_e^* be the subgroup in the isotropy group $H^* \simeq \text{Ad}(H)$ given by $\text{Ad}(h)e = e$, $h \in H$, generated by the Lie subalgebra $\mathfrak{h}_{\parallel} \subset \mathfrak{h}$. The subgroup H_e^* can be identified with the group $\text{U}(n-1, \mathbb{H}) \times \text{U}(1, \mathbb{H}) \subset \text{U}(n+1, \mathbb{H})$ whose matrix representation is given by

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & A \end{pmatrix}, \quad a \in \text{U}(1, \mathbb{H}), \quad A \in \text{U}(n-1, \mathbb{H}). \quad (2.29)$$

Here the subgroup $\text{U}(n-1, \mathbb{H})$ acts on \mathfrak{m} by right multiplication,

$$\text{Ad}(A)\mathfrak{m}_{\parallel} = \mathfrak{m}_{\parallel}, \quad \text{Ad}(A)\mathfrak{m}_{\perp} = \mathfrak{m}_{\perp}, \quad \text{Ad}(A)\mathbf{m}_{\perp} = \mathbf{m}_{\perp}A, \quad (2.30)$$

while $\text{U}(1, \mathbb{H})$ has a non-standard action given by

$$\text{Ad}(a)\mathfrak{m}_{\parallel} = \mathfrak{m}_{\parallel}, \quad \text{Ad}(a)\mathfrak{m}_{\perp} = a\mathfrak{m}_{\perp}a^{-1}, \quad \text{Ad}(a)\mathbf{m}_{\perp} = a\mathbf{m}_{\perp}. \quad (2.31)$$

The corresponding group action on \mathfrak{h} is given by

$$\text{Ad}(a)\mathfrak{h}_{\parallel} = a\mathfrak{h}_{\parallel}a^{-1}, \quad \text{Ad}(a)\mathbf{H}_{\parallel} = \mathbf{H}_{\parallel}, \quad \text{Ad}(a)\mathfrak{h}_{\perp} = a\mathfrak{h}_{\perp}a^{-1}, \quad \text{Ad}(a)\mathbf{h}_{\perp} = a\mathbf{h}_{\perp}. \quad (2.32)$$

Proposition 2.4. *The vector subspaces \mathfrak{h}_{\perp} and \mathfrak{m}_{\perp} are isomorphic under the linear map $\text{ad}(e)$, and consequently they each decompose into a direct sum of vector subspaces given by irreducible representations of the group H_e^* on which the linear map $\text{ad}(e)^2$ is a multiple of the identity:*

$$\text{ad}(e)^2(\mathfrak{m}_{\perp}, \mathbf{m}_{\perp}) = (-4\mathfrak{m}_{\perp}, -\mathbf{m}_{\perp}) \in \mathfrak{m}_{\perp}, \quad \text{ad}(e)^2(\mathfrak{h}_{\perp}, \mathbf{h}_{\perp}) = (-4\mathfrak{h}_{\perp}, -\mathbf{h}_{\perp}) \in \mathfrak{h}_{\perp}. \quad (2.33)$$

The Lie bracket relations on $\mathfrak{m} = \mathfrak{m}_\parallel \oplus \mathfrak{m}_\perp$ and $\mathfrak{h} = \mathfrak{h}_\parallel \oplus \mathfrak{h}_\perp$ coming from the structure of \mathfrak{g} as a symmetric Lie algebra (2.16) consist of

$$[\mathfrak{m}_\parallel, \mathfrak{m}_\parallel] \subseteq \mathfrak{h}_\parallel, \quad [\mathfrak{m}_\parallel, \mathfrak{h}_\parallel] \subseteq \mathfrak{m}_\parallel, \quad [\mathfrak{h}_\parallel, \mathfrak{h}_\parallel] \subseteq \mathfrak{h}_\parallel, \quad (2.34)$$

$$[\mathfrak{m}_\parallel, \mathfrak{m}_\perp] \subseteq \mathfrak{h}_\perp, \quad [\mathfrak{m}_\parallel, \mathfrak{h}_\perp] \subseteq \mathfrak{m}_\perp, \quad [\mathfrak{h}_\parallel, \mathfrak{m}_\perp] \subseteq \mathfrak{m}_\perp, \quad [\mathfrak{h}_\parallel, \mathfrak{h}_\perp] \subseteq \mathfrak{h}_\perp. \quad (2.35)$$

The only Lie brackets with nontrivial decompositions are $[\mathfrak{m}_\perp, \mathfrak{m}_\perp]$, $[\mathfrak{h}_\perp, \mathfrak{h}_\perp]$, $[\mathfrak{m}_\perp, \mathfrak{h}_\perp]$. To write out all these brackets explicitly, we introduce the following commutator and anti-commutator notations. For $\mathbf{a}, \mathbf{b} \in \mathbb{Q}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{H}^{n-1}$, let

$$C(\mathbf{a}, \mathbf{b}) := \mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a} \in \mathbb{Q}, \quad A(\mathbf{a}, \mathbf{b}) := \mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a} \in \mathbb{R}, \quad (2.36a)$$

$$C(\mathbf{a}, \mathbf{b}) := \langle \mathbf{a}, \mathbf{b} \rangle - \langle \mathbf{b}, \mathbf{a} \rangle = \mathbf{a}\bar{\mathbf{b}}^t - \mathbf{b}\bar{\mathbf{a}}^t \in \mathbb{Q}, \quad \mathbf{C}(\mathbf{a}, \mathbf{b}) = \bar{\mathbf{a}}^t \mathbf{b} - \bar{\mathbf{b}}^t \mathbf{a} \in \mathfrak{u}(n-1, \mathbb{H}), \quad (2.36b)$$

$$A(\mathbf{a}, \mathbf{b}) := \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{a} \rangle = \mathbf{a}\bar{\mathbf{b}}^t + \mathbf{b}\bar{\mathbf{a}}^t \in \mathbb{R}. \quad (2.36c)$$

Lemma 2.5.

1. The Lie brackets (2.34), (2.35) are given by

$$[(\mathbf{m}_{1\parallel}), (\mathbf{m}_{2\parallel})] = 0 \in \mathfrak{h}_\parallel, \quad (2.37a)$$

$$[(\mathbf{m}_\parallel), (\mathbf{h}_\parallel, \mathbf{H}_\parallel)] = 0 \in \mathfrak{m}_\parallel, \quad (2.37b)$$

$$[(\mathbf{h}_{1\parallel}, \mathbf{H}_{1\parallel}), (\mathbf{h}_{2\parallel}, \mathbf{H}_{2\parallel})] = (C(\mathbf{h}_{1\parallel}, \mathbf{h}_{2\parallel}), [\mathbf{H}_{1\parallel}, \mathbf{H}_{2\parallel}]) \in \mathfrak{h}_\parallel, \quad (2.37c)$$

$$[(\mathbf{m}_\parallel), (\mathbf{m}_\perp, \mathbf{m}_\perp)] = (2\mathbf{m}_\parallel \mathbf{m}_\perp, -\mathbf{m}_\parallel \mathbf{m}_\perp) \in \mathfrak{h}_\perp, \quad (2.37d)$$

$$[(\mathbf{m}_\parallel), (\mathbf{h}_\perp, \mathbf{h}_\perp)] = (-2\mathbf{m}_\parallel \mathbf{h}_\perp, \mathbf{m}_\parallel \mathbf{h}_\perp) \in \mathfrak{m}_\perp, \quad (2.37e)$$

$$[(\mathbf{h}_\parallel, \mathbf{H}_\parallel), (\mathbf{m}_\perp, \mathbf{m}_\perp)] = (C(\mathbf{h}_\parallel, \mathbf{m}_\perp), \mathbf{h}_\parallel \mathbf{m}_\perp - \mathbf{m}_\perp \mathbf{H}_\parallel) \in \mathfrak{m}_\perp, \quad (2.37f)$$

$$[(\mathbf{h}_\parallel, \mathbf{H}_\parallel), (\mathbf{h}_\perp, \mathbf{h}_\perp)] = (C(\mathbf{h}_\parallel, \mathbf{h}_\perp), \mathbf{h}_\parallel \mathbf{h}_\perp - \mathbf{h}_\perp \mathbf{H}_\parallel) \in \mathfrak{h}_\perp. \quad (2.37g)$$

2. The remaining Lie brackets are given by the projections

$$[(\mathbf{m}_{1\perp}, \mathbf{m}_{1\perp}), (\mathbf{m}_{2\perp}, \mathbf{m}_{2\perp})]_{\mathfrak{h}_\parallel} = (C(\mathbf{m}_{1\perp}, \mathbf{m}_{2\perp}) + \frac{1}{2}C(\mathbf{m}_{2\perp}, \mathbf{m}_{1\perp}), \mathbf{C}(\mathbf{m}_{2\perp}, \mathbf{m}_{1\perp})) \in \mathfrak{h}_\parallel, \quad (2.38a)$$

$$[(\mathbf{m}_{1\perp}, \mathbf{m}_{1\perp}), (\mathbf{m}_{2\perp}, \mathbf{m}_{2\perp})]_{\mathfrak{h}_\perp} = (\frac{1}{2}C(\mathbf{m}_{2\perp}, \mathbf{m}_{1\perp}), \mathbf{m}_{1\perp} \mathbf{m}_{2\perp} - \mathbf{m}_{2\perp} \mathbf{m}_{1\perp}) \in \mathfrak{h}_\perp, \quad (2.38b)$$

$$[(\mathbf{h}_{1\perp}, \mathbf{h}_{1\perp}), (\mathbf{h}_{2\perp}, \mathbf{h}_{2\perp})]_{\mathfrak{h}_\parallel} = (C(\mathbf{h}_{1\perp}, \mathbf{h}_{2\perp}) - \frac{1}{2}C(\mathbf{h}_{2\perp}, \mathbf{h}_{1\perp}), \mathbf{C}(\mathbf{h}_{2\perp}, \mathbf{h}_{1\perp})) \in \mathfrak{h}_\parallel, \quad (2.38c)$$

$$[(\mathbf{h}_{1\perp}, \mathbf{h}_{1\perp}), (\mathbf{h}_{2\perp}, \mathbf{h}_{2\perp})]_{\mathfrak{h}_\perp} = (\frac{1}{2}C(\mathbf{h}_{1\perp}, \mathbf{h}_{2\perp}), \mathbf{h}_{2\perp} \mathbf{h}_{1\perp} - \mathbf{h}_{1\perp} \mathbf{h}_{2\perp}) \in \mathfrak{h}_\perp, \quad (2.38d)$$

$$[(\mathbf{m}_\perp, \mathbf{m}_\perp), (\mathbf{h}_\perp, \mathbf{h}_\perp)]_{\mathfrak{m}_\parallel} = (-A(\mathbf{m}_\perp, \mathbf{h}_\perp) - \frac{1}{2}A(\mathbf{m}_\perp, \mathbf{h}_\perp)) \in \mathfrak{m}_\parallel, \quad (2.38e)$$

$$[(\mathbf{m}_\perp, \mathbf{m}_\perp), (\mathbf{h}_\perp, \mathbf{h}_\perp)]_{\mathfrak{m}_\perp} = (\frac{1}{2}C(\mathbf{h}_\perp, \mathbf{m}_\perp), \mathbf{m}_\perp \mathbf{h}_\perp - \mathbf{h}_\perp \mathbf{m}_\perp) \in \mathfrak{m}_\perp. \quad (2.38f)$$

3. The Cartan-Killing form on $\mathfrak{h}_\perp \simeq \mathfrak{m}_\perp$ is given by

$$\langle (\mathbf{h}_{1\perp}, \mathbf{h}_{1\perp}), (\mathbf{h}_{2\perp}, \mathbf{h}_{2\perp}) \rangle = \chi \text{Re}(\mathbf{h}_{1\perp} \mathbf{h}_{2\perp} - \mathbf{h}_{1\perp} \bar{\mathbf{h}}_{2\perp}^t) = \frac{\chi}{2}(A(\mathbf{h}_{1\perp}, \mathbf{h}_{2\perp}) - A(\mathbf{h}_{1\perp}, \mathbf{h}_{2\perp})). \quad (2.39)$$

An explicit basis for \mathfrak{m} can be built from the basis $\{1, i, j, k\}$ for $\mathbb{R} \oplus \mathbb{Q}$ together with the standard orthonormal basis $\{\mathbf{e}_l\}_{l=1, \dots, n-1}$ of \mathbb{R}^{n-1} as follows:

$$\begin{aligned} \mathfrak{m}_\parallel &= \text{span}(\mathbf{e}) \simeq \mathbb{R}, \\ \mathfrak{m}_\perp &= \text{span}(\mathbf{m}_\perp^i, \mathbf{m}_\perp^j, \mathbf{m}_\perp^k) \oplus \text{span}(\mathbf{m}_{\perp l}^i, \mathbf{m}_{\perp l}^j, \mathbf{m}_{\perp l}^k)_{l=1, \dots, n-1} \simeq \mathbb{Q} \oplus \mathbb{H}^{n-1} \end{aligned} \quad (2.40)$$

in which

$$\begin{aligned}\mathbf{m}_{\perp l} = (0, \mathbf{e}_l) &= \begin{pmatrix} 0 & 0 & \mathbf{e}_l \\ 0 & 0 & \mathbf{0} \\ -\mathbf{e}_l^t & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{m}_{\perp l}^q = (0, q\mathbf{e}_l) = \begin{pmatrix} 0 & 0 & q\mathbf{e}_l \\ 0 & 0 & \mathbf{0} \\ \mathbf{e}_l^t q & \mathbf{0} & \mathbf{0} \end{pmatrix}, \\ \mathbf{m}_{\perp}^q = (q, \mathbf{0}) &= \begin{pmatrix} 0 & q & \mathbf{0} \\ q & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},\end{aligned}\tag{2.41}$$

where $q = i, j, k$, and $l = 1, \dots, n-1$. Note $\mathbf{m}_{\perp l}^q$ and \mathbf{m}_{\perp}^q are, respectively, related to $\mathbf{m}_{\perp l}$ and \mathbf{e} by the action of $U(1, \mathbb{H}) \subset U(n+1, \mathbb{H})$ defined by $\text{Ad}(a)\mathbf{e} = \mathbf{m}_{\perp}^q$, $\text{Ad}(a)\mathbf{m}_{\perp l} = \mathbf{m}_{\perp l}^q$ where

$$a = \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}, \quad q = i, j, k \in \mathbb{Q}.\tag{2.42}$$

In addition, $\mathbf{m}_{\perp l}$ is itself related to \mathbf{e} by the action of the rotation subgroup $SO(n, \mathbb{R}) \subset U(n, \mathbb{H}) \subset U(n+1, \mathbb{H})$.

Finally, we record several useful quaternionic identities connected with properties of the Cartan-Killing form (2.39).

(1) For $a \in \mathbb{Q}$, $b, c \in \mathbb{H}$ and $\mathbf{b}, \mathbf{c} \in \mathbb{H}^{n-1}$:

$$\overline{\langle b, c \rangle} = \langle c, b \rangle, \quad \text{Re}(a \langle b, c \rangle) = \text{Re}(\langle b, c \rangle a) = -\text{Re}(a \langle c, b \rangle),\tag{2.43}$$

$$\overline{\langle \mathbf{b}, \mathbf{c} \rangle} = \langle \mathbf{c}, \mathbf{b} \rangle, \quad \text{Re}(a \langle \mathbf{b}, \mathbf{c} \rangle) = \text{Re}(\langle \mathbf{b}, \mathbf{c} \rangle a) = -\text{Re}(a \langle \mathbf{c}, \mathbf{b} \rangle).\tag{2.44}$$

(2) For $a, b, c \in \mathbb{Q}$:

$$\text{Re}(ab) = \text{Re}(ba), \quad \text{Re}(abc) = \text{Re}(bca) = \text{Re}(cab) = -\text{Re}(bac) = -\text{Re}(cba) = -\text{Re}(acb).\tag{2.45}$$

3. FRAME FORMULATION OF NON-STRETCHING CURVE FLOWS IN \mathbb{HP}^n

We consider non-stretching flows of framed curves in the quaternionic projective space $M = \mathbb{HP}^n$ viewed as a symmetric Riemannian geometry (2.11). Its Riemannian structure will be described in terms of [25] a \mathfrak{m} -valued linear coframe e and a \mathfrak{h} -valued linear connection ω whose torsion and curvature

$$\mathfrak{T} := de + [\omega, e], \quad \mathfrak{R} := d\omega + \frac{1}{2}[\omega, \omega]\tag{3.1}$$

are 2-forms with respective values in \mathfrak{m} and \mathfrak{h} , given by the following Cartan structure equations:

$$\mathfrak{T} = 0, \quad \mathfrak{R} = -\frac{1}{2}[e, e].\tag{3.2}$$

Here $[\cdot, \cdot]$ denotes the Lie bracket on $\mathfrak{g} = \mathfrak{u}(n+1, \mathbb{H})$ composed with the wedge product on $T_x^*M \simeq \mathbb{H}^n$. This structure together with the (negative-definite) Cartan-Killing form determines a Riemannian metric and Riemannian connection (i.e. covariant derivative) on the manifold M as follows. For all X, Y in $T_x M$,

$$g(X, Y) := -\langle e_X, e_Y \rangle, \quad e \rfloor \nabla_X Y := d_X e_Y + [\omega_X, e_Y],\tag{3.3}$$

where the coframe provides an identification between the tangent space $T_x M$ and the vector space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ as given by $e \rfloor X := e_X$, $e \rfloor Y := e_Y \in \mathfrak{m}$. The connection is metric compatible,

$\nabla g = 0$, and torsion-free, $T = 0$, while its curvature is covariantly constant, $\nabla R = 0$. In particular,

$$e \rfloor R(X, Y)Z = [\mathfrak{R}](X \wedge Y, e_Z) = -[[e_X, e_Y], e_Z], \quad e \rfloor T(X, Y) = \mathfrak{T}(X \wedge Y) = 0, \quad (3.4)$$

where $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is the torsion tensor and $R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ is the curvature tensor. Note the linear coframe and linear connection have gauge freedom given by the following transformations

$$e \longrightarrow \text{Ad}(h^{-1})e, \quad \omega \longrightarrow \text{Ad}(h^{-1})\omega + h^{-1}dh \quad (3.5)$$

for an arbitrary function $h : M \rightarrow H = \text{U}(n, \mathbb{H}) \times \text{U}(1, \mathbb{H}) \subset \text{U}(n+1, \mathbb{H})$. These gauge transformations comprise a local (x -dependent) representation of the linear isotropy group $H^* = \text{Ad}(H)$ which defines the gauge group [26] of the frame bundle of $M = \mathbb{H}\mathbb{P}^n$. Both the metric tensor g and curvature tensor R on $M = \mathbb{H}\mathbb{P}^n$ are gauge invariant.

Let $\gamma(t, x)$ be a flow of any smooth curve in $M = \mathbb{H}\mathbb{P}^n$. We write $X = \gamma_x$ for the tangent vector and $Y = \gamma_t$ for the evolution vector at each point x along the curve. Note the flow is *non-stretching* provided that it preserves the $\text{U}(n+1, \mathbb{H})$ -invariant arclength $ds = |\gamma_x|dx$, or equivalently $\nabla_t |\gamma_x| = 0$, in which case we can put

$$|\gamma_x| = 1 \quad (3.6)$$

without loss of generality. For smooth flows that are transverse to the curve (such that X and Y are linearly independent), $\gamma(t, x)$ will describe a smooth two-dimensional surface in M . The pullback of the Cartan structure equations (3.2) to this surface yields a framing of the curve evolving under the flow:

$$D_x e_t - D_t e_x + [\omega_x, e_t] - [\omega_t, e_x] = 0, \quad (3.7)$$

$$D_x \omega_t - D_t \omega_x + [\omega_x, \omega_t] = -[e_x, e_t], \quad (3.8)$$

with

$$e_x := e \rfloor X = e \rfloor \gamma_x, \quad e_t := e \rfloor Y = e \rfloor \gamma_t, \quad (3.9)$$

$$\omega_x := \omega \rfloor X = \omega \rfloor \gamma_x, \quad \omega_t := \omega \rfloor Y = \omega \rfloor \gamma_t, \quad (3.10)$$

where D_x, D_t denote total derivatives with respect to x, t .

Remarkably, for any non-stretching curve flow, these torsion and curvature equations (3.7)–(3.10) encode an explicit bi-Hamiltonian structure once a specific choice of frame along $\gamma(t, x)$ is made [22]. A frame consists of a set of orthonormal vectors that span the tangent space of M at each point x on the curve. Associated to a frame is the connection matrix consisting of the set of frame components of the covariant x -derivative of each frame vector along the curve [27]. Such a framing is obtained from the Lie-algebra components of e^* and ω_x when an orthonormal basis is introduced for \mathfrak{m} and \mathfrak{h} with respect to the Cartan-Killing form, where e^* is a \mathfrak{m} -valued linear frame defined to be dual to the coframe e by the condition that $-\langle e^*, e \rangle = \text{id}$ is the identity map on each tangent space $T_\gamma M$ (cf [22, 25]). In particular, ω_x determines e along the curve via the transport equation

$$\nabla_x e = -\text{ad}(\omega_x)e. \quad (3.11)$$

Then if $\{\mathbf{m}_l\}_{l=1, \dots, 4n-1}$ is any fixed orthonormal basis for the perp space of $\text{span}(e_x)$ in \mathfrak{m} , the set of vectors given by $X_{\perp l} := -\langle e^*, \mathbf{m}_l \rangle$ together with $X = -\langle e^*, e_x \rangle$ defines a frame at each point x along the curve, i.e. $\text{span}(X_{\perp 1}, \dots, X_{\perp 4n-1}, X) = T_\gamma M$ with the subset $\{X_{\perp l}\}_{l=1, \dots, 4n-1}$ being an orthonormal basis for the normal space of the curve. It turns

out that the bi-Hamiltonian structure encoded in the resulting frame structure equations for $\gamma(t, x)$ is independent of the choice of a basis, and accordingly the simplest algebraic formulation of this encoding arises in terms of the \mathfrak{m} -valued coframe and \mathfrak{h} -valued connection variables (3.9) and (3.10).

We utilize a natural choice of a moving frame defined by the following two properties which are a direct algebraic generalization of a parallel moving frame in Euclidean geometry.

- (1) e_x is a constant unit-norm element lying in a fixed Cartan subspace $\mathfrak{a} \subset \mathfrak{m}$ (with $\dim(\mathfrak{a}) = \text{rank}(\mathfrak{m}) = 1$), i.e. $D_x e_x = D_t e_x = 0$, $\langle e_x, e_x \rangle = -1$, and $\mathfrak{a} = \text{span}(e_x) = \mathfrak{m}_{\parallel}$ where $\mathfrak{m}_{\parallel} \oplus \mathfrak{m}_{\perp} = \mathfrak{m}$ and $\langle \mathfrak{m}_{\parallel}, \mathfrak{m}_{\perp} \rangle = 0$.
- (2) ω_x lies in the perp space \mathfrak{h}_{\perp} of the Lie subalgebra $\mathfrak{h}_{\parallel} \subset \mathfrak{h}$ of the linear isotropy group $H_{\parallel}^* \subset H^* \simeq \text{Ad}(U(1, \mathbb{H}) \times U(n, \mathbb{H}))$ that preserves e_x , i.e. $\text{ad}(\mathfrak{h}_{\parallel})e_x = 0$ and $\langle \omega_x, \mathfrak{h}_{\parallel} \rangle = 0$ where $\mathfrak{h}_{\parallel} \oplus \mathfrak{h}_{\perp} = \mathfrak{h}$ and $\langle \mathfrak{h}_{\parallel}, \mathfrak{h}_{\perp} \rangle = 0$.

Such a moving frame for $\gamma(t, x)$ will be called $U(1, \mathbb{H}) \times U(n, \mathbb{H})$ -parallel and its existence can be established by constructing a suitable gauge transformation (3.5) on an arbitrary moving frame at each point x along the curve [22]. Since every Cartan subspace $\mathfrak{a} \subset \mathfrak{m}$ is one-dimensional, a $U(1, \mathbb{H}) \times U(n, \mathbb{H})$ -parallel moving frame is unique up to the rigid (x -independent) action of H_{\parallel}^* on e and ω_x . Specifically, given any \mathfrak{m} -valued linear coframe \tilde{e} and \mathfrak{h} -valued linear connection matrix $\tilde{\omega}_x$ along γ , we can first find a gauge transformation such that $h^{-1}\tilde{e}_x h = e_x$ is a constant element in any Cartan subspace $\mathfrak{a} \subset \mathfrak{m}$, as a consequence of the fact $\mathfrak{m} = \text{Ad}(H)\mathfrak{a}$. The norm of e_x will satisfy $-\langle e_x, e_x \rangle = g(\gamma_x, \gamma_x) = |\gamma_x|^2 = 1$ because we have chosen an arclength parametrization of the curve. We can then find a gauge transformation belonging to the subgroup H_{\parallel}^* which preserves e_x , so that $h^{-1}D_x h + h^{-1}\tilde{\omega}_x h = \omega_x$ where $h(x) \in H_{\parallel}^*$ is given by solving the linear matrix ODE $D_x h + \tilde{\omega}^{\parallel} h = 0$ in terms of the decomposition of $\tilde{\omega}_x = \tilde{\omega}^{\parallel} + \tilde{\omega}^{\perp}$ relative to e_x . Note the solution will depend on an arbitrary initial condition $h(x_0) \in H_{\parallel}^*$, specified at some point $x = x_0$ along the curve, which represents a rigid gauge freedom (i.e. the equivalence group) in the construction of the moving frame.

Employing the quaternionic matrix notation and algebraic preliminaries from Section 2, we choose

$$\begin{aligned} e_x &= \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & 1 & \mathbf{0} \\ -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} =: \frac{1}{\sqrt{\chi}}(1, \mathbf{0}) \in \mathfrak{a} \simeq \mathbb{R}, \\ \omega_x &= \begin{pmatrix} \mathbf{u} & 0 & \mathbf{0} \\ 0 & -\mathbf{u} & \mathbf{u} \\ \mathbf{0} & -\bar{\mathbf{u}}^t & \mathbf{0} \end{pmatrix} =: (\mathbf{u}, \mathbf{u}) \in \mathfrak{h}_{\perp} \simeq \mathbb{Q} \oplus \mathbb{H}^{n-1}, \end{aligned} \tag{3.12}$$

where $\mathbf{u} \in \mathbb{Q}$ is an imaginary quaternion variable, and $\mathbf{u} \in \mathbb{H}^{n-1}$ is a quaternionic vector variable. (Note the factor $1/\sqrt{\chi}$ in e_x comes from the normalization factor (2.21) of the Cartan-Killing inner product on \mathfrak{m} .) The equivalence group H_{\parallel}^* of the $U(1, \mathbb{H}) \times U(n, \mathbb{H})$ -parallel frame obtained from this choice of e_x and ω_x consists of rigid (x -independent) gauge transformations (3.5) that preserve the form of the matrices (3.12). Specifically, we have from (2.31) and (2.32), $\text{Ad}(h^{-1})(1, \mathbf{0}) = (1, \mathbf{0})$ and $\text{Ad}(h^{-1})(\mathbf{u}, \mathbf{u}) = (\tilde{\mathbf{u}}, \tilde{\mathbf{u}})$ where

$$\tilde{\mathbf{u}} = a \mathbf{u} a^{-1} \in \mathbb{Q} \quad \text{and} \quad \tilde{\mathbf{u}} = a \mathbf{u} A \in \mathbb{H}^{n-1} \tag{3.13}$$

for all constant functions (on M) $h = (a, A) \in H_{\parallel} = \mathrm{U}(1, \mathbb{H}) \times \mathrm{U}(n-1, \mathbb{H})$ with the explicit matrix form (2.29).

Hereafter e will denote the \mathfrak{m} -valued linear coframe along $\gamma(t, x)$ as determined (up to equivalence) by (3.12) through the transport condition (3.11). In terms of the dual frame e^* , the associated framing for $T_{\gamma}M$ will look like

$$\begin{aligned} X &:= -\langle e^*, e_x \rangle = \gamma_x, & X_{\perp l} &:= -\langle e^*, \mathbf{m}_{\perp l} \rangle \perp \gamma_x, \\ X_{\perp}^q &:= -\langle e^*, \mathbf{m}_{\perp}^q \rangle \perp \gamma_x, & X_{\perp l}^q &:= -\langle e^*, \mathbf{m}_{\perp l}^q \rangle \perp \gamma_x, \quad q = i, j, k \in \mathbb{Q} \end{aligned} \quad (3.14)$$

where $\{\mathbf{m}_{\perp}^q, \mathbf{m}_{\perp l}^q, \mathbf{m}_{\perp l}^q\}_{l=1, \dots, n-1}$ give an orthonormal basis for \mathfrak{m}_{\perp} while $\{e_x\}$ is a unit basis for \mathfrak{m}_{\parallel} whence $\{X, X_{\perp}^q, X_{\perp l}^q, X_{\perp l}^q\}_{l=1, \dots, n-1}$ is an orthonormal frame along $\gamma(t, x)$ such that $dx = ds$ is the $\mathrm{U}(n+1, \mathbb{H})$ -invariant arclength. An explicit choice of such a basis for \mathfrak{m} is shown in (2.41). The action of $\mathrm{Ad}(H_{\parallel}) \simeq H_{\parallel}^*$ on this basis will yield all other choices of an orthonormal basis for \mathfrak{m} as given by (2.30)–(2.31).

The tangent space of M along $\gamma(t, x)$ thereby has an orthogonal decomposition into real and imaginary subspaces $T_{\gamma}M \simeq \mathbb{R}^n \oplus \mathbb{Q}^n$ defined by

$$\mathbb{R}^n = \mathrm{span}(X, X_{\perp l})_{l=1, \dots, n-1} =: \mathrm{Re}T_{\gamma}M, \quad \mathbb{Q}^n = \mathrm{span}(X_{\perp}^q, X_{\perp l}^q)_{l=1, \dots, n-1}^{q=i, j, k} =: \mathrm{Im}T_{\gamma}M.$$

In terms of this structure, the $\mathrm{U}(1, \mathbb{H}) \times \mathrm{U}(n, \mathbb{H})$ -parallel frame (3.14) has the geometrical property

$$\mathrm{Re}\nabla_x X \in \mathrm{span}(X_{\perp 1}, \dots, X_{\perp n-1}) \quad \text{and} \quad \mathrm{Re}\nabla_x X_{\perp l} \in \mathrm{span}(X), \quad l = 1, \dots, n-1, \quad (3.15)$$

closely resembling a Euclidean parallel frame [20].

The geometrical meaning of the $\mathrm{U}(1, \mathbb{H}) \times \mathrm{U}(n, \mathbb{H})$ -parallel connection determined by (3.14) is seen from looking at the frame components of the principal normal vector

$$N := \nabla_x X = \langle e^*, \mathrm{ad}(e_x)\omega_x \rangle \quad (3.16)$$

given by

$$e \lrcorner N = -\mathrm{ad}(e_x)\omega_x = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & 2\mathbf{u} & -\mathbf{u} \\ 2\mathbf{u} & 0 & \mathbf{0} \\ \overline{\mathbf{u}}^t & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathfrak{m}_{\perp}. \quad (3.17)$$

These components \mathbf{u} and \mathbf{u} are invariantly defined by $\gamma(t, x)$ up to the rigid (x -independent) action of the equivalence group $H_{\parallel}^* \simeq \mathrm{Ad}(\mathrm{U}(1, \mathbb{H}) \times \mathrm{U}(n-1, \mathbb{H}))$ that preserves the framing at each point x . Hence, in geometrical terms, the scalar-vector pair (\mathbf{u}, \mathbf{u}) describes *covariants* of the curve γ relative to the group H_{\parallel}^* , such that each component $\mathbf{u} \in \mathbb{Q}$, $\mathbf{u} \in \mathbb{H}^{n-1}$ belongs to an irreducible representation of this group corresponding to the orthogonal decomposition of the vector space $\mathfrak{m}_{\perp} \simeq \mathbb{Q} \oplus \mathbb{H}^{n-1}$. Moreover, x -derivatives of (\mathbf{u}, \mathbf{u}) describe differential covariants of γ relative to H_{\parallel}^* , which arise geometrically from the frame components of ∇_x -derivatives of the principal normal vector N . For example, $(\mathbf{u}_x, \mathbf{u}_x)$ corresponds to

$$\begin{aligned} e \lrcorner \nabla_x N &= -\mathrm{ad}(e_x)D_x\omega_x + \mathrm{ad}(\omega_x)^2 e_x \\ &= \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & 2\mathbf{u}_x + 4\mathbf{u}^2 - |\mathbf{u}|^2 & -\mathbf{u}_x - 3\mathbf{u}\mathbf{u} \\ 2\mathbf{u}_x - 4\mathbf{u}^2 + |\mathbf{u}|^2 & 0 & \mathbf{0} \\ \overline{\mathbf{u}}_x^t - 3\overline{\mathbf{u}}^t \mathbf{u} & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathfrak{m}_{\perp}. \end{aligned} \quad (3.18)$$

We thus note that the geometric invariants of γ as defined by Riemannian inner products of the tangent vector $X = \gamma_x$ and its derivatives $N = \nabla_x \gamma_x$, $\nabla_x N = \nabla_x^2 \gamma_x$, etc. along the curve

γ can be expressed as scalars formed from Cartan-Killing inner products of the covariants (\mathbf{u}, \mathbf{u}) and differential covariants $(\mathbf{u}_x, \mathbf{u}_x), (\mathbf{u}_{2x}, \mathbf{u}_{2x})$, etc.; in particular,

$$g(N, N) = -g(X, \nabla_x^2 X) = 4|\mathbf{u}|^2 + |\mathbf{u}|^2, \quad (3.19)$$

$$g(N, \nabla_x N) = -g(X, \nabla_x^3 X) = 4\text{Re}\langle \mathbf{u}, \mathbf{u}_x \rangle + \text{Re}\langle \mathbf{u}, \mathbf{u}_x \rangle, \quad (3.20)$$

$$g(\nabla_x N, \nabla_x N) = g(X, \nabla_x^4 X) = 4|\mathbf{u}_x|^2 + |\mathbf{u}_x|^2 + (4|\mathbf{u}|^2 + |\mathbf{u}|^2)^2 + 9|\mathbf{u}|^2|\mathbf{u}|^2 + 6\text{Re}\langle \mathbf{u}\mathbf{u}, \mathbf{u}_x \rangle \quad (3.21)$$

comprise all $U(n+1, \mathbb{H})$ -invariants depending on at most $\mathbf{u}, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_x$.

Remark 3.1. *The set of $4n-1$ invariants given by $\{g(X, \nabla_x^l X)\}_{l=2, \dots, 4n}$ (and their x -derivatives up to differential order $4n-l$) generate the components of the connection matrix of a classical Frenét frame [27] determined by γ_x .*

4. BI-HAMILTONIAN STRUCTURE

Let $\gamma(t, x)$ be any non-stretching curve flow in $M = \mathbb{HP}^n$ and choose a $U(1, \mathbb{H}) \times U(n, \mathbb{H})$ -parallel framing given by (3.12). In the flow direction $Y = \gamma_t$, we decompose $e_t = h_{\parallel} + h_{\perp}$ relative to e_x , with

$$h_{\parallel} = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & h_{\parallel} & \mathbf{0} \\ -h_{\parallel} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{pmatrix} =: \frac{1}{\sqrt{\chi}}(h_{\parallel}) \in \mathfrak{m}_{\parallel} \simeq \mathbb{R}, \quad (4.1)$$

$$h_{\perp} = \begin{pmatrix} 0 & h_{\perp} & \mathbf{h}_{\perp} \\ h_{\perp} & 0 & \mathbf{0} \\ -\bar{\mathbf{h}}_{\perp}^t & \mathbf{0} & 0 \end{pmatrix} =: (h_{\perp}, \mathbf{h}_{\perp}) \in \mathfrak{m}_{\perp} \simeq \mathbb{Q} \oplus \mathbb{H}^{n-1}, \quad (4.2)$$

and likewise for $\omega_t = \varpi^{\parallel} + \varpi^{\perp}$ with

$$\varpi^{\parallel} = \begin{pmatrix} \mathbf{w}^{\parallel} & 0 & \mathbf{0} \\ 0 & \mathbf{w}^{\parallel} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{W}^{\parallel} \end{pmatrix} =: (\mathbf{w}^{\parallel}, \mathbf{W}^{\parallel}) \in \mathfrak{h}_{\parallel} \simeq \mathbb{Q} \oplus \mathfrak{u}(n-1, \mathbb{H}), \quad (4.3)$$

$$\varpi^{\perp} = \begin{pmatrix} \mathbf{w}^{\perp} & 0 & \mathbf{0} \\ 0 & -\mathbf{w}^{\perp} & \mathbf{w}^{\perp} \\ \mathbf{0} & -\bar{\mathbf{w}}^{\perp t} & \mathbf{0} \end{pmatrix} =: (\mathbf{w}^{\perp}, \mathbf{w}^{\perp}) \in \mathfrak{h}_{\perp} \simeq \mathbb{Q} \oplus \mathbb{H}^{n-1}, \quad (4.4)$$

where $h_{\parallel} \in \mathbb{R}$ is a real scalar variable, $h_{\perp}, \mathbf{w}^{\parallel}, \mathbf{w}^{\perp} \in \mathbb{Q}$ are imaginary quaternion variables, $\mathbf{h}_{\perp}, \mathbf{w}^{\perp} \in \mathbb{H}^{n-1}$ are quaternion vector variables, and $\mathbf{W}^{\parallel} \in \mathfrak{u}(n-1, \mathbb{H})$ is a quaternion unitary matrix. Here we have inserted a factor $1/\sqrt{\chi}$ in h_{\parallel} , which corresponds to the normalization factor in e_x , in order to simplify later expressions.

The frame formulation of the flow $\gamma(t, x)$ is then given by the general results in [22] as follows.

Lemma 4.1. *The Cartan structure equations (3.7), (3.8) for any $U(1, \mathbb{H}) \times U(n, \mathbb{H})$ -parallel linear coframe e and linear connection ω pulled back to the two-dimensional surface $\gamma(t, x)$ take the form of a flow on $\omega_x =: u(t, x)$ given by*

$$u_t = D_x \varpi^{\perp} + [u, \varpi^{\parallel}] + [u, \varpi^{\perp}]_{\perp} + \text{ad}(e_x)h_{\perp}, \quad (4.5)$$

where

$$\varpi^\perp = -\text{ad}(e_x)^{-1}(D_x h_\perp + [u, h_\parallel] + [u, h_\perp]_\perp), \quad (4.6)$$

$$h_\parallel = -D_x^{-1}[u, h_\perp]_\parallel, \quad \varpi^\parallel = -D_x^{-1}[u, \varpi^\perp]_\parallel, \quad (4.7)$$

are given in terms of h_\perp , with D_x^{-1} denoting the formal inverse of the total x -derivative operator D_x .

Note that h_\perp geometrically corresponds to the normal part of the flow vector Y ,

$$h_\perp = e \rfloor Y_\perp$$

where Y_\perp is the orthogonal projection of Y relative to the tangent vector X along the curve. Similarly, the tangential part of the flow vector Y corresponds to h_\parallel , which is related to h_\perp through (4.7) as a geometrical consequence of the non-stretching property of the flow, $\nabla_t |X| = 0$, and the torsion-free property of the framing, $\nabla_t X = \nabla_x Y$.

We thus emphasize that the formulation stated in Lemma 4.1 applies to *all* non-stretching curve flows $\gamma(t, x)$ in $\mathbb{H}\mathbb{P}^n$, with the flow being determined by specifying Y_\perp freely as a function of t at each point x along the curve.

We now proceed with presenting the bi-Hamiltonian operators encoded in the Cartan structure equations (4.5), (4.6), (4.7). Write

$$h^\perp := \text{ad}(e_x)h_\perp = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 2h_\perp & 0 & \mathbf{0} \\ 0 & -2h_\perp & -\mathbf{h}_\perp \\ \mathbf{0} & \bar{\mathbf{h}}^t & \mathbf{0} \end{pmatrix} := (h^\perp, \mathbf{h}^\perp) \in \mathfrak{h}_\perp \quad (4.8)$$

which belongs to the same space as $u = (\mathbf{u}, \mathbf{u})$, where

$$\sqrt{\chi}h^\perp = 2h_\perp \in \mathbb{Q}, \quad \sqrt{\chi}\mathbf{h}^\perp = -\mathbf{h}_\perp \in \mathbb{H}^{n-1}. \quad (4.9)$$

Then the flow equation (4.5) can be written in operator form [22]

$$u_t = \mathcal{H}(\varpi^\perp) + h^\perp, \quad \varpi^\perp = \mathcal{J}(h^\perp), \quad (4.10)$$

where

$$\mathcal{H} = \mathcal{K}|_{\mathfrak{h}_\perp}, \quad \mathcal{J} = -\text{ad}(e_x)^{-1}\mathcal{K}|_{\mathfrak{m}_\perp}\text{ad}(e_x)^{-1} \quad (4.11)$$

are linear operators which act on \mathfrak{h}_\perp -valued functions and are invariant under H_\parallel^* , as defined in terms of the operator

$$\mathcal{K} := D_x + [u, \cdot]_\perp - [u, D_x^{-1}[u, \cdot]_\parallel]. \quad (4.12)$$

To display these operators explicitly, we first write out the flow on the scalar and vector variables $u = (\mathbf{u}, \mathbf{u})$, yielding

$$\mathbf{u}_t = D_x \mathbf{w}^\perp + C(\mathbf{u}, \mathbf{w}^\parallel) + \frac{1}{2}C(\mathbf{u}, \mathbf{w}^\perp) + \mathbf{h}^\perp, \quad (4.13)$$

$$\mathbf{u}_t = D_x \mathbf{w}^\perp - \mathbf{w}^\parallel \mathbf{u} + \mathbf{u} \mathbf{W}^\parallel + \mathbf{w}^\perp \mathbf{u} - \mathbf{u} \mathbf{w}^\perp + \mathbf{h}^\perp, \quad (4.14)$$

where

$$\mathbf{w}^\parallel = -D_x^{-1}(C(\mathbf{u}, \mathbf{w}^\perp) - \frac{1}{2}C(\mathbf{u}, \mathbf{w}^\perp)), \quad \mathbf{W}^\parallel = D_x^{-1}C(\mathbf{u}, \mathbf{w}^\perp), \quad (4.15)$$

$$\mathbf{w}^\perp = \chi \left(\frac{1}{4}D_x h^\perp + \frac{1}{4}C(\mathbf{u}, \mathbf{h}^\perp) \right) + h_\parallel \mathbf{u}, \quad \mathbf{w}^\perp = \chi \left(D_x \mathbf{h}^\perp + \frac{1}{2}h^\perp \mathbf{u} + \mathbf{u} h^\perp \right) + h_\parallel \mathbf{u}, \quad (4.16)$$

$$h_\parallel = -\chi D_x^{-1} \left(\frac{1}{2}A(\mathbf{u}, \mathbf{h}^\perp) - \frac{1}{2}A(\mathbf{u}, \mathbf{h}^\perp) \right). \quad (4.17)$$

Next we introduce some operator notations. For $\mathbf{v} \in \mathbb{Q}$, $\mathbf{v} \in \mathbb{H}^{n-1}$, $\mathbf{V} \in \mathfrak{u}(n-1, \mathbb{H})$, let

$$\begin{aligned} C_{\mathbf{u}}\mathbf{v} &:= C(\mathbf{u}, \mathbf{v}) \in \mathbb{Q}, & C_{\mathbf{u}}\mathbf{v} &:= \frac{1}{2}C(\mathbf{u}, \mathbf{v}) \in \mathbb{Q}, \\ \mathbf{C}_{\mathbf{u}}\mathbf{v} &:= \mathbf{C}(\mathbf{u}, \mathbf{v}) \in \mathfrak{u}(n-1, \mathbb{H}), & A_{\mathbf{u}}\mathbf{v} &:= \frac{1}{2}A(\mathbf{u}, \mathbf{v}) \in \mathbb{R} \end{aligned} \quad (4.18)$$

denoting commutator and anti-commutator operators, and let

$$R_{\mathbf{u}}\mathbf{v} := \mathbf{v}\mathbf{u} \in \mathbb{H}^{n-1}, \quad L_{\mathbf{u}}\mathbf{v} := \mathbf{u}\mathbf{v} \in \mathbb{H}^{n-1}, \quad L_{\mathbf{u}}\mathbf{V} := \mathbf{u}\mathbf{V} \in \mathbb{H}^{n-1} \quad (4.19)$$

denoting right and left multiplication. Last it is convenient to scale the variables

$$\mathbf{h}^\perp \rightarrow \frac{1}{\chi}\mathbf{h}^\perp, \quad \mathbf{h}^\perp \rightarrow \frac{1}{\chi}\mathbf{h}^\perp \quad (4.20)$$

in order to absorb all χ factors in (4.16)–(4.17).

Theorem 4.2. *The scaled flow equations (4.13), (4.14), (4.20) for the quaternion variables $\mathbf{u}(t, x) \in \mathbb{Q}$ and $\mathbf{u}(t, x) \in \mathbb{H}^{n-1}$ have the operator form*

$$\begin{pmatrix} \mathbf{u}_t \\ \mathbf{w}^\perp \end{pmatrix} = \mathcal{H} \begin{pmatrix} \mathbf{w}^\perp \\ \mathbf{h}^\perp \end{pmatrix} + \chi^{-1} \begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{h}^\perp \end{pmatrix}, \quad \begin{pmatrix} \mathbf{w}^\perp \\ \mathbf{w}^\perp \end{pmatrix} = \mathcal{J} \begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{h}^\perp \end{pmatrix}, \quad (4.21)$$

where

$$\mathcal{H} = \begin{pmatrix} D_x - C_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}} & C_{\mathbf{u}} + C_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}} \\ R_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}} + R_{\mathbf{u}} & D_x - R_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}} + L_{\mathbf{u}}D_x^{-1}\mathbf{C}_{\mathbf{u}} - L_{\mathbf{u}} \end{pmatrix} \quad (4.22)$$

and

$$\mathcal{J} = \begin{pmatrix} \frac{1}{4}D_x - \frac{1}{4}A_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} & \frac{1}{2}C_{\mathbf{u}} + \frac{1}{2}A_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} \\ \frac{1}{2}R_{\mathbf{u}} - \frac{1}{2}R_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} & D_x + L_{\mathbf{u}} + R_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} \end{pmatrix} \quad (4.23)$$

are compatible Hamiltonian cosymplectic and symplectic operators on the x -jet space of (\mathbf{u}, \mathbf{u}) .

The proof of Theorem 4.2 follows directly from general results proven in [22] on the Hamiltonian structure of non-stretching curve flows in symmetric spaces, as applied to $\mathbb{HP}^n \simeq \mathrm{Sp}(n+1)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$. The work in [22] also develops the basic theory and properties of bi-Hamiltonian operators for Lie-algebra valued flow variables, generalizing the standard treatment for scalar variables given in [28] (see also [29]).

To explain the definition and properties of Hamiltonian operators in the setting of quaternions, we start by defining variational derivatives on the x -jet space $J^\infty := (x, \mathbf{u}, \mathbf{u}, \bar{\mathbf{u}}, \mathbf{u}_x, \mathbf{u}_x, \bar{\mathbf{u}}_x, \dots)$ of the quaternionic flow variables $\mathbf{u}(t, x)$ and $\bar{\mathbf{u}}(t, x)$. For any real-valued functional $\mathfrak{H} = \int H(x, \mathbf{u}, \mathbf{u}, \bar{\mathbf{u}}, \mathbf{u}_x, \mathbf{u}_x, \bar{\mathbf{u}}_x, \dots) dx$, its variational derivatives with respect to \mathbf{u} and $\bar{\mathbf{u}}$ are defined in terms of Frechet derivatives of $H(x, \mathbf{u}, \mathbf{u}, \bar{\mathbf{u}}, \mathbf{u}_x, \mathbf{u}_x, \bar{\mathbf{u}}_x, \dots)$ by

$$\delta_{\mathbf{h}^\perp} \mathfrak{H} = \int \mathrm{pr}(\mathbf{h}^\perp \cdot \partial/\partial \mathbf{u}) H dx =: \int \mathrm{Re}(\mathbf{h}^\perp (\delta \mathfrak{H} / \delta \mathbf{u})) dx = \mathrm{Re} \int \langle \mathbf{h}^\perp, -\delta \mathfrak{H} / \delta \mathbf{u} \rangle dx, \quad (4.24)$$

$$\delta_{\mathbf{h}^\perp} \mathfrak{H} = \int \mathrm{pr}(\mathrm{Re}(\mathbf{h}^\perp \cdot \partial/\partial \mathbf{u})) H dx =: \int \mathrm{Re}(\mathbf{h}^\perp (\delta \mathfrak{H} / \delta \mathbf{u})^\dagger) = \mathrm{Re} \int \langle \mathbf{h}^\perp, \delta \mathfrak{H} / \delta \bar{\mathbf{u}} \rangle dx \quad (4.25)$$

modulo total x -derivatives, holding for all imaginary quaternion functions \mathbf{h}^\perp and all quaternionic vector functions \mathbf{h}^\perp , where $\mathbf{h}^\perp \cdot \partial/\partial \mathbf{u}$ and $\mathrm{Re}(\mathbf{h}^\perp \cdot \partial/\partial \mathbf{u}) = \frac{1}{2}(\mathbf{h}^\perp \cdot \partial/\partial \mathbf{u} + \bar{\mathbf{h}}^\perp \cdot \partial/\partial \bar{\mathbf{u}})$ are

corresponding vector fields prolonged to J^∞ . (Here the dot denotes summation over quaternion components.) We note these definitions of $\delta\mathfrak{H}/\delta\mathbf{u}$ and $\delta\mathfrak{H}/\delta\mathbf{u}$ involve a reordering of products of quaternion variables (as given by the middle equalities of (4.24) and (4.25)), for which we use the quaternionic multiplication and conjugation identities (2.43)–(2.45).

Now the property stated in Theorem 4.2 that the operator \mathcal{H} is cosymplectic means it defines an associated Poisson bracket

$$\{\mathfrak{H}_1, \mathfrak{H}_2\} := \text{Re} \int \left\langle \begin{pmatrix} -\delta\mathfrak{H}_1/\delta\mathbf{u} \\ \delta\mathfrak{H}_1/\delta\bar{\mathbf{u}} \end{pmatrix}, \mathcal{H} \begin{pmatrix} -\delta\mathfrak{H}_2/\delta\mathbf{u} \\ \delta\mathfrak{H}_2/\delta\bar{\mathbf{u}} \end{pmatrix} \right\rangle dx \quad (4.26)$$

which is skew-symmetric and obeys the Jacobi identity, for all real-valued functionals $\mathfrak{H}_1, \mathfrak{H}_2$. A counterpart of the Poisson bracket is the symplectic 2-form defined in terms of the operator \mathcal{J} by

$$\omega(X_1, X_2)_{\mathcal{J}} := \text{Re} \int \left\langle \begin{pmatrix} X_1\mathbf{u} \\ X_1\mathbf{u} \end{pmatrix}, \mathcal{J} \begin{pmatrix} X_2\mathbf{u} \\ X_2\mathbf{u} \end{pmatrix} \right\rangle dx \quad (4.27)$$

where X_1, X_2 are vector fields $X = \mathbf{h}^\perp \cdot \partial/\partial\mathbf{u} + \text{Re}(\mathbf{h}^\perp \cdot \partial/\partial\mathbf{u})$ associated to pairs of an imaginary quaternion function \mathbf{h}^\perp and a quaternion vector function \mathbf{h}^\perp . The property stated in Theorem 4.2 that \mathcal{J} is symplectic corresponds to ω being skew-symmetric and closed. In particular, closure means that

$$\begin{aligned} 0 &= \text{pr}(X_1)\omega(X_2, X_3) + \text{cyclic} \\ &= \text{Re} \int \left\langle \begin{pmatrix} \mathbf{h}_2^\perp \\ \mathbf{h}_2^\perp \end{pmatrix}, \text{pr}(\mathbf{h}_1^\perp \cdot \partial/\partial\mathbf{u} + \text{Re}(\mathbf{h}_1^\perp \cdot \partial/\partial\mathbf{u}))\mathcal{J} \begin{pmatrix} \mathbf{h}_3^\perp \\ \mathbf{h}_3^\perp \end{pmatrix} \right\rangle dx + \text{cyclic} \end{aligned} \quad (4.28)$$

holds modulo total x -derivatives for all vector fields X_1, X_2, X_3 . Compatibility of these operators \mathcal{H} and \mathcal{J} is the statement that every linear combination $c_1\mathcal{H} + c_2\mathcal{J}^{-1}$ is a cosymplectic Hamiltonian operator, or equivalently that $c_1\mathcal{H}^{-1} + c_2\mathcal{J}$ is a symplectic operator, where \mathcal{H}^{-1} and \mathcal{J}^{-1} denote formal inverse operators defined on the x -jet space J^∞ .

5. BI-HAMILTONIAN HIERARCHIES OF SOLITON EQUATIONS

Composition of the compatible Hamiltonian operators (4.22) and (4.23) yields a recursion operator

$$\mathcal{R} := \mathcal{H}\mathcal{J} = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix} \quad (5.1)$$

given by

$$\begin{aligned} \mathcal{R}_{11} &= \frac{1}{4}D_x^2 + \frac{1}{2}C_{\mathbf{u}}R_{\mathbf{u}} - \frac{1}{4}D_xA_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} - \frac{1}{4}C_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}D_x + \frac{1}{2}C_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}R_{\mathbf{u}}, \\ \mathcal{R}_{12} &= \frac{1}{2}D_xC_{\mathbf{u}} + C_{\mathbf{u}}D_x + C_{\mathbf{u}}L_{\mathbf{u}} + \frac{1}{2}D_xA_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} + C_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}D_x - \frac{1}{2}C_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}C_{\mathbf{u}} \\ &\quad + C_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}L_{\mathbf{u}}, \\ \mathcal{R}_{21} &= \frac{1}{2}D_xR_{\mathbf{u}} + \frac{1}{4}R_{\mathbf{u}}D_x - \frac{1}{2}L_{\mathbf{u}}R_{\mathbf{u}} + \frac{1}{4}R_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}D_x - \frac{1}{2}D_xR_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} - \frac{1}{4}R_{\mathbf{u}}A_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} \\ &\quad - \frac{1}{2}R_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}R_{\mathbf{u}} + \frac{1}{2}L_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}R_{\mathbf{u}} + \frac{1}{2}L_{\mathbf{u}}R_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}}, \\ \mathcal{R}_{22} &= D_x^2 + D_xL_{\mathbf{u}} - L_{\mathbf{u}}D_x - L_{\mathbf{u}}L_{\mathbf{u}} + \frac{1}{2}R_{\mathbf{u}}C_{\mathbf{u}} + D_xR_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} - R_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}D_x + L_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}D_x \\ &\quad + \frac{1}{2}R_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}C_{\mathbf{u}} + \frac{1}{2}R_{\mathbf{u}}A_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}} - R_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}L_{\mathbf{u}} + L_{\mathbf{u}}D_x^{-1}C_{\mathbf{u}}L_{\mathbf{u}} - L_{\mathbf{u}}R_{\mathbf{u}}D_x^{-1}A_{\mathbf{u}}. \end{aligned}$$

Each of these operators $\mathcal{H}, \mathcal{J}, \mathcal{R}$ displays obvious symmetry invariance under translations in x . As a consequence, from general results due to Magri [30, 31], the recursion operator

will generate a hierarchy of commuting Hamiltonian vector fields with respect to the Poisson bracket, starting from the evolutionary form of the x -translation vector field $\partial/\partial x$. Moreover, the adjoint recursion operator \mathcal{R}^* will generate an involutive hierarchy of variational covector fields, arising from the canonical pairing provided by the symplectic 2-form. This leads to the following results [22].

Theorem 5.1. *The pairs of quaternionic scalar-vector functions $\mathbf{h}_{(l)}^\perp \in \mathbb{Q}$, $\mathbf{h}_{(l)}^\perp \in \mathbb{H}^{n-1}$ given by*

$$\begin{pmatrix} \mathbf{h}_{(l)}^\perp \\ \mathbf{h}_{(l)}^\perp \end{pmatrix} := \mathcal{R}^l \begin{pmatrix} \mathbf{u}_x \\ \mathbf{u}_x \end{pmatrix}, \quad l = 0, 1, 2, \dots, \quad (5.2)$$

yield a commuting hierarchy of Hamiltonian vector fields $\mathbf{h}_{(l)}^\perp \cdot \partial/\partial \mathbf{u} + \text{Re}(\mathbf{h}_{(l)}^\perp \cdot \partial/\partial \mathbf{u})$. In particular, there exists corresponding Hamiltonian functionals $\mathfrak{H}^{(l)}$ such that

$$\delta_{(\mathbf{h}_{(l)}^\perp, \mathbf{h}_{(l)}^\perp)} \mathfrak{G} = \{\mathfrak{G}, \mathfrak{H}^{(l)}\}_{\mathcal{H}}$$

for all functionals \mathfrak{G} on J^∞ . Explicit expressions for the Hamiltonians $H^{(l)} \in \mathbb{R}$, $l = 0, 1, 2, \dots$, are given by

$$\begin{aligned} H^{(l)} &:= \frac{1}{1+2l} D_x^{-1} \text{Re}(\langle \mathbf{u}, \mathbf{h}_{(l)}^\perp \rangle + \langle \mathbf{u}, \mathbf{h}_{(l)}^\perp \rangle) \\ &= \frac{1}{2+4l} D_x^{-1} (-A(\mathbf{u}, \mathbf{h}_{(l)}^\perp) + A(\mathbf{u}, \mathbf{h}_{(l)}^\perp)) = \frac{1}{1+2l} \mathbf{h}_{(l)}^\perp \end{aligned} \quad (5.3)$$

whose variational derivatives

$$\mathbf{w}_{(l)}^\perp := \delta H^{(l)} / \delta \bar{\mathbf{u}} = -\delta H^{(l)} / \delta \mathbf{u} \in \mathbb{Q}, \quad \mathbf{w}_{(l)}^\perp := \delta H^{(l)} / \delta \bar{\mathbf{u}} \in \mathbb{H}^{n-1}$$

yield an associated hierarchy of involutive covector fields $\mathbf{w}_{(l)}^\perp \cdot d\mathbf{u} + \text{Re}(\mathbf{w}_{(l)}^\perp \cdot d\mathbf{u})$ given by

$$\begin{pmatrix} \mathbf{w}_{(l)}^\perp \\ \mathbf{w}_{(l)}^\perp \end{pmatrix} := \mathcal{R}^{*l} \begin{pmatrix} \mathbf{u} \\ \mathbf{u} \end{pmatrix}, \quad l = 0, 1, 2, \dots \quad (5.4)$$

These variational covector fields are dual to the Hamiltonian vector fields via the pairing

$$(\mathbf{w}_{(l)}^\perp \cdot d\mathbf{u} + \text{Re}(\mathbf{w}_{(l)}^\perp \cdot d\mathbf{u})) \rfloor X = \boldsymbol{\omega}(X, \mathbf{h}_{(l)}^\perp \cdot \partial/\partial \mathbf{u} + \text{Re}(\mathbf{h}_{(l)}^\perp \cdot \partial/\partial \mathbf{u})) \quad (5.5)$$

holding for all vector fields X in evolutionary form on J^∞ .

Note both hierarchies (5.2) and (5.4) possess the mKdV scaling symmetry $x \rightarrow \lambda x$, $(\mathbf{u}, \mathbf{u}) \rightarrow (\lambda^{-1} \mathbf{u}, \lambda^{-1} \mathbf{u})$, with (5.2) and (5.3) having the scaling weight $2 + 2l$ and (5.4) having the scaling weight $1 + 2l$.

We can now state our main result.

Theorem 5.2. *The flow equations (4.21) on the imaginary scalar quaternion variable $\mathbf{u}(t, x)$ and the $n-1$ -component quaternion vector variable $\mathbf{u}(t, x)$ yield a hierarchy of bi-Hamiltonian evolution equations*

$$\begin{pmatrix} \mathbf{u}_t \\ \mathbf{u}_t \end{pmatrix} = \begin{pmatrix} \mathbf{h}_{(l)}^\perp + \chi^{-1} \mathbf{h}_{(l-1)}^\perp \\ \mathbf{h}_{(l)}^\perp + \chi^{-1} \mathbf{h}_{(l-1)}^\perp \end{pmatrix}, \quad l = 1, 2, \dots, \quad (5.6)$$

called the $(+l)$ -flow, with the Hamiltonian structure

$$\begin{pmatrix} \mathbf{h}_{(l)}^\perp \\ \mathbf{h}_{(l)}^\perp \end{pmatrix} = \mathcal{H} \begin{pmatrix} -\delta H^{(l)}/\delta \mathbf{u} \\ \delta H^{(l)}/\delta \bar{\mathbf{u}} \end{pmatrix}, \quad l = 0, 1, 2, \dots, \quad (5.7)$$

$$= \mathcal{E} \begin{pmatrix} -\delta H^{(l-1)}/\delta \mathbf{u} \\ \delta H^{(l-1)}/\delta \bar{\mathbf{u}} \end{pmatrix}, \quad l = 1, 2, \dots, \quad (5.8)$$

where $\mathcal{E} := \mathcal{RH}$. Each of these multi-component quaternionic evolution equations (5.6) is invariant under the group $H_\parallel^* \simeq \text{Ad}(\text{U}(1, \mathbb{H}) \times \text{U}(n-1, \mathbb{H}))$ given by rigid (x -independent) transformations (3.13) on the pair of scalar-vector variables (\mathbf{u}, \mathbf{u}) .

5.1. mKdV flow. The $+1$ flow in the hierarchy (5.6) comes from

$$\begin{pmatrix} \mathbf{h}_{(0)}^\perp \\ \mathbf{h}_{(0)}^\perp \end{pmatrix} = \begin{pmatrix} \mathbf{u}_x \\ \mathbf{u}_x \end{pmatrix} \quad (5.9)$$

producing a coupled system of quaternionic scalar-vector mKdV equations

$$\mathbf{u}_t - \chi^{-1} \mathbf{u}_x = \frac{1}{4} \mathbf{u}_{3x} - \frac{3}{2} \mathbf{u}^2 \mathbf{u}_x + \frac{3}{4} C(\mathbf{u}, C(\mathbf{u}, \mathbf{u}_x)) + \frac{3}{4} C(\mathbf{u}, \mathbf{u}_{2x}) = \mathbf{h}_{(1)}^\perp, \quad (5.10)$$

$$\mathbf{u}_t - \chi^{-1} \mathbf{u}_x = \mathbf{u}_{3x} + \frac{3}{2} (|\mathbf{u}|^2 - \mathbf{u}^2 + \mathbf{u}_x) \mathbf{u}_x + \frac{3}{4} (2\mathbf{u}|\mathbf{u}|^2 - A(\mathbf{u}, \mathbf{u}_x) + \mathbf{u}_{2x} - C(\mathbf{u}, \mathbf{u}_x)) \mathbf{u} = \mathbf{h}_{(1)}^\perp, \quad (5.11)$$

where $C(\cdot, \cdot)$ and $A(\cdot, \cdot)$ respectively denote the commutators and anticommutators defined in (2.36a)–(2.36c).

This is an integrable system in the following sense. We first note that the convective terms \mathbf{u}_x and \mathbf{u}_x in both equations (5.10)–(5.11) can be removed by a Galilean transformation $t \rightarrow t, x \rightarrow x + \chi^{-1}t$. The resulting evolution equations then have the explicit bi-Hamiltonian structure given by the $l = 1$ case of (5.7)–(5.8):

$$\begin{aligned} \begin{pmatrix} \mathbf{u}_t \\ \mathbf{u}_t \end{pmatrix} &= \mathcal{H} \begin{pmatrix} \delta H^{(1)}/\delta \mathbf{u} \\ \delta H^{(1)}/\delta \bar{\mathbf{u}} \end{pmatrix} = \mathcal{E} \begin{pmatrix} \delta H^{(0)}/\delta \mathbf{u} \\ \delta H^{(0)}/\delta \bar{\mathbf{u}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} \mathbf{u}_{3x} - \frac{3}{2} \mathbf{u}^2 \mathbf{u}_x + \frac{3}{4} C(\mathbf{u}, C(\mathbf{u}, \mathbf{u}_x)) + \frac{3}{4} C(\mathbf{u}, \mathbf{u}_{2x}) \\ \mathbf{u}_{3x} + \frac{3}{2} (|\mathbf{u}|^2 - \mathbf{u}^2 + \mathbf{u}_x) \mathbf{u}_x + \frac{3}{4} (2\mathbf{u}|\mathbf{u}|^2 - A(\mathbf{u}, \mathbf{u}_x) - C(\mathbf{u}, \mathbf{u}_x) + \mathbf{u}_{2x}) \mathbf{u} \end{pmatrix} \end{aligned} \quad (5.12)$$

where

$$H^{(0)} = -\frac{1}{2} \mathbf{u}^2 + |\mathbf{u}|^2 \quad \text{and} \quad H^{(1)} = \frac{1}{8} \mathbf{u}_x^2 - \frac{1}{2} |\mathbf{u}_x|^2 - \frac{1}{8} A(\mathbf{u}, C(\mathbf{u}, \mathbf{u}_x)) + \frac{1}{8} (\mathbf{u}^2 - |\mathbf{u}|^2)^2 \quad (5.13)$$

are the Hamiltonians.

In addition to its explicit symmetry with respect to space translations $x \rightarrow x + \epsilon$ and mKdV scalings $x \rightarrow \lambda x, t \rightarrow \lambda^3 t, \mathbf{u} \rightarrow \lambda^{-1} \mathbf{u}, \bar{\mathbf{u}} \rightarrow \lambda^{-1} \bar{\mathbf{u}}$, the quaternion scalar-vector mKdV system (5.12) possesses higher symmetries given by each higher-order flow in the hierarchy, i.e. $\mathbf{X} = \mathbf{h}_{(l)}^\perp \cdot \partial/\partial \mathbf{u} + \text{Re}(\mathbf{h}_{(l)}^\perp \cdot \partial/\partial \bar{\mathbf{u}})$ generates an infinitesimal symmetry of the coupled system (5.12) for all $l = 2, 3, \dots$. Moreover, each Hamiltonian in the hierarchy yields a conservation law for this system (5.12), i.e. $\frac{d}{dt} \int_{-\infty}^{\infty} H^{(l)} dx = 0$ for $l = 0, 1, 2, \dots$ holds for all solutions $\mathbf{u}(t, x), \bar{\mathbf{u}}(t, x)$ that have sufficiently fast decay as $x \rightarrow \pm\infty$.

5.2. SG flow. Apart from the $+1, +2, \dots$ flows in the mKdV hierarchy (5.6), the recursion operator $\mathcal{R} = \mathcal{H}\mathcal{J}$ also yields a flow defined by

$$0 = \begin{pmatrix} \mathbf{w}^\perp \\ \mathbf{w}^\perp \end{pmatrix} = \mathcal{J} \begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{h}^\perp \end{pmatrix}. \quad (5.14)$$

This will be called the -1 flow [22]. The resulting flow equations (4.21) have the form

$$\begin{pmatrix} \mathbf{u}_t \\ \mathbf{u}_t \end{pmatrix} = \chi^{-1} \begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{u}^\perp \end{pmatrix} \quad (5.15)$$

with

$$D_x \mathbf{h}^\perp = -C(\mathbf{u}, \mathbf{h}^\perp) - 4\mathbf{h}_\parallel \mathbf{u}, \quad D_x \mathbf{h}^\perp = -\frac{1}{2}\mathbf{h}^\perp \mathbf{u} - \mathbf{u} \mathbf{h}^\perp - \mathbf{h}_\parallel \mathbf{u} \quad (5.16)$$

and

$$D_x \mathbf{h}_\parallel = -\frac{1}{2}A(\mathbf{u}, \mathbf{h}^\perp) - \frac{1}{2}A(\mathbf{u}, \mathbf{h}^\perp). \quad (5.17)$$

These equations (5.16)–(5.17) possess the conservation law

$$D_x (\mathbf{h}_\parallel^2 + \frac{1}{4}|\mathbf{h}^\perp|^2 + |\mathbf{h}^\perp|^2) = 0. \quad (5.18)$$

Hence, after a conformal scaling of t , which induces a corresponding scaling of the variables $\mathbf{h}_\parallel, \mathbf{h}^\perp, \mathbf{h}^\perp$ by a function of t , we get

$$\mathbf{h}_\parallel^2 + \frac{1}{4}|\mathbf{h}^\perp|^2 + |\mathbf{h}^\perp|^2 = c = \text{const.} \quad (5.19)$$

yielding the relation

$$\mathbf{h}_\parallel = \pm \sqrt{c - \frac{1}{4}|\mathbf{h}^\perp|^2 - |\mathbf{h}^\perp|^2}. \quad (5.20)$$

Substitution of (5.20) and (5.15) into (5.16) then gives a coupled hyperbolic system of quaternionic scalar-vector SG equations

$$\mathbf{u}_{tx} = \mp 4\sqrt{(1 - \frac{1}{4}|\mathbf{u}_t|^2 - |\mathbf{u}_t|^2)} \mathbf{u} - C(\mathbf{u}, \mathbf{u}_t), \quad (5.21)$$

$$\mathbf{u}_{tx} = \mp \sqrt{(1 - \frac{1}{4}|\mathbf{u}_t|^2 - |\mathbf{u}_t|^2)} \mathbf{u} - \frac{1}{2}\mathbf{u}_t \mathbf{u} - \mathbf{u} \mathbf{u}_t, \quad (5.22)$$

in which we have put $c = \chi^2$ without loss of generality and scaled out a factor χ by means of the transformation $x \rightarrow \chi^{-1}x$. Here $C(\cdot, \cdot)$ denotes the commutator defined in (2.36b).

This system (5.21)–(5.22) has explicit symmetry under separate time translations $t \rightarrow t + \epsilon$ and space translations $x \rightarrow x + \epsilon$ as well as scalings $t \rightarrow \lambda^{-1}t$, $\mathbf{u} \rightarrow \lambda^{-1}\mathbf{u}$, $\mathbf{u} \rightarrow \lambda^{-1}\mathbf{u}$. It also possesses higher symmetries $X = \mathbf{h}_{(l)}^\perp \cdot \partial/\partial \mathbf{u} + \text{Re}(\mathbf{h}_{(l)}^\perp \cdot \partial/\partial \mathbf{u})$ for $l = 2, 3, \dots$, given by each higher-order flow (5.2) in the quaternionic mKdV hierarchy (5.2), along with conservation laws $\frac{d}{dt} \int_{-\infty}^{\infty} H^{(l)} dx = 0$ for $l = 0, 1, 2, \dots$, given by the corresponding Hamiltonians (5.3) in the same hierarchy, as verified through (5.15)–(5.17).

5.3. Reductions and Soliton equations in $\mathbb{H}P^1$. We now make some remarks on reductions of the quaternionic mKdV system (5.12) and quaternionic SG system (5.21).

Firstly, if we consider a reduction to a vector system by putting $\mathbf{u} = 0$, then the quaternion vector inner product terms $C(\mathbf{u}, \mathbf{u}_{2x})$ and $C(\mathbf{u}, \mathbf{u}_t)$ in the respective equations (5.12) and (5.21) for the quaternion scalar variable \mathbf{u} need to vanish identically. Such algebraic constraints cannot be satisfied unless we restrict the vector variable \mathbf{u} to be commutative by taking its components to belong to a real or complex subalgebra of \mathbb{H} . Therefore, both the quaternion mKdV system (5.12) and quaternion SG system (5.21) have *no* consistent

non-commutative vector reduction. The underlying reason why the quaternion vector variable \mathbf{u} is unavoidably coupled to the quaternion scalar variable u can be understood from the nontrivial Lie bracket relation $[(0, \mathbf{h}_{1\perp}), (0, \mathbf{h}_{2\perp})]_{\parallel} = (\frac{1}{2}C(\mathbf{h}_{1\perp}, \mathbf{h}_{2\perp}), \mathbf{0})$ for the space \mathfrak{h}_{\perp} in which \mathbf{u} lies (cf. (2.38d) and (3.12)).

In contrast, we can get a consistent scalar reduction by putting $\mathbf{u} = 0$ in the systems (5.12) and (5.21). This reduction has a natural geometric meaning if non-stretching curve flows are considered in a submanifold $\mathbb{HP}^1 \subset \mathbb{HP}^n$ or in \mathbb{HP}^1 itself when $n = 1$.

Thereby we obtain a scalar non-commutative mKdV equation

$$u_t = \frac{1}{4}u_{3x} - \frac{3}{2}u^2u_x \quad (5.23)$$

and a scalar non-commutative SG equation

$$u_{tx} = 4u\sqrt{1 + \frac{1}{4}u_t^2} \quad (5.24)$$

in which $u(t, x)$ is an imaginary quaternion variable. We note the underlying Hamiltonian structure in this case is given by Theorems 5.1 and 5.2 with $\mathbf{u} = 0$ and $\mathbf{h}^{\perp} = 0$, where the Hamiltonian operators take form

$$\mathcal{H} = D_x - C_u D_x^{-1} C_u, \quad \mathcal{J} = \frac{1}{4}D_x - \frac{1}{4}A_u D_x^{-1} A_u \quad (5.25)$$

(which are the first diagonal entries of (4.22) and (4.23), respectively).

6. GEOMETRIC CURVE FLOWS

The bi-Hamiltonian flows given in Theorems 5.1 and 5.2 have a direct geometrical formulation in terms of the $U(1, \mathbb{H}) \times U(n, \mathbb{H})$ -parallel frame variables (3.9), (3.10), (3.12) describing a non-stretching flow of a curve γ in $M = \mathbb{HP}^n$. To begin, we recall that the quaternionic components of the connection matrix $\omega_x = (u, \mathbf{u}) = u$ describe covariants of γ relative to the equivalence group of the framing. This means that u is invariantly determined by γ up to the action of the group

$$H_{\parallel}^* \simeq \text{Ad}(U(1, \mathbb{H}) \times U(n-1, \mathbb{H})) \quad (6.1)$$

as given by the rigid (x -independent) transformations (3.13). The evolution of u under the bi-Hamiltonian flows (5.6)–(5.8) is given by

$$u_t = h_{(l+1)}^{\perp} + \chi^{-1} h_{(l)}^{\perp} \quad (6.2)$$

with

$$h_{(l)}^{\perp} = \mathcal{R}^l u_x \quad (6.3)$$

for all $l = 0, 1, 2, \dots$, where \mathcal{R} is the recursion operator defined in terms of u through (4.11) and (4.12). Each evolution equation (6.2) is invariant with respect to the transformations (3.13) and belongs to the general class of flows in which

$$h^{\perp}(x, u, u_x, u_{xx}, \dots) = \text{Ad}(a^{-1})h^{\perp}(x, \text{Ad}(a)u, \text{Ad}(a)u_x, \text{Ad}(a)u_{xx}, \dots), \quad (6.4)$$

$$a \in U(1, \mathbb{H}) \times U(n-1, \mathbb{H})$$

is an equivariant function of the invariant arclength x of γ and the (differential) covariants u, u_x, u_{xx}, \dots of γ relative to the group (6.1).

Any flow specified by the $U(1, \mathbb{H}) \times U(n-1, \mathbb{H})$ -equivariant class (6.4) yields a non-stretching curve flow in $M = \mathbb{HP}^n$ via the geometric relations

$$h_{\perp} = \chi^{-1} \text{ad}(e_x)^{-1} h^{\perp} = e \rfloor Y_{\perp}, \quad h_{\parallel} = -D_x^{-1}[u, h_{\perp}]_{\parallel} = e \rfloor Y_{\parallel} \quad (6.5)$$

taking into account (4.8) and (4.20), where Y_\perp and Y_\parallel are the normal and tangential projections of $Y = \gamma_t$ relative to the tangent vector $X = \gamma_x$ along γ . In particular, the evolution vector of the curve is given by

$$Y = -\langle e^*, h_\perp + h_\parallel \rangle = \chi^{-1} \langle e^*, \mathcal{Y}(h^\perp) \rangle \quad (6.6)$$

in terms of the operator

$$\mathcal{Y} := D_x^{-1}[u, \text{ad}(e_x)^{-1} \cdot]_\parallel - \text{ad}(e_x)^{-1} \quad (6.7)$$

where e^* is the linear frame dual to the linear coframe e along γ , with $e_x = e \rfloor X$. In this relation (6.6), e_x is preserved under the action of the equivalence group (6.1), while up to equivalence, both e^* and e are determined by ω_x through the transport equation (3.11) along γ . Hence we obtain the following geometric characterization of the class of flows (6.4).

Proposition 6.1. *Every $U(1, \mathbb{H}) \times U(n-1, \mathbb{H})$ -equivariant flow (6.4) determines a non-stretching curve flow $\gamma(t, x)$ in $\mathbb{H}\mathbb{P}^n$ satisfying a $U(n+1, \mathbb{H})$ -invariant evolution equation*

$$\gamma_t = Y(x, \gamma_x, \nabla_x \gamma_x, \nabla_x^2 \gamma_x, \dots) \quad (6.8)$$

through (6.6)–(6.7), where x is the $U(n+1, \mathbb{H})$ -invariant arclength along the curve.

Invariance of an evolution equation (6.8) can be shown to imply that the function Y is constructed using only the metric g given by (3.3) and the tensor ad_x^2 defined as follows:

$$e \rfloor \text{ad}_x^2(X)Z := \text{ad}(e_X)^2 e_Z \quad (6.9)$$

for all X, Z in $T_x M$. Note both g and ad_x^2 are gauge-invariant under changes of frame (3.5) and thus are well-defined as geometrical structures on the manifold $M = \mathbb{H}\mathbb{P}^n$.

We can now formulate the bi-Hamiltonian flows from theorem 5.1 and 5.2 in strictly geometrical terms.

Theorem 6.2. *The hierarchy of bi-Hamiltonian quaternionic flows (5.6)–(5.8) correspond to non-stretching geometric curve flows in $M = \mathbb{H}\mathbb{P}^n$ given by evolution equations of the form*

$$\gamma_t = Y_{(l)}(\gamma_x, \nabla_x \gamma_x, \nabla_x^2 \gamma_x, \dots), \quad |\gamma_x| = 1, \quad (l = 1, 2, \dots)$$

where each equation is invariant with respect to the isometry group $U(n+1, \mathbb{H})$ of $M = \mathbb{H}\mathbb{P}^n$ and preserves the invariant arclength x .

To write out these geometric curve flow equations explicitly, it is useful to introduce the linear map

$$\mathcal{X}_\gamma := -\text{ad}_x^2(\gamma_x) \quad (6.10)$$

which is determined by the curve γ . In a $U(1, \mathbb{H}) \times U(n-1, \mathbb{H})$ -parallel frame, this map corresponds to $-\text{ad}(e_x)^2$ under which the vector space $\mathfrak{m} = \mathfrak{u}(n+1, \mathbb{H})/(\mathfrak{u}(1, \mathbb{H}) \oplus \mathfrak{u}(n, \mathbb{H}))$ decomposes into a direct sum of eigenspaces $\mathfrak{m}_\parallel = \mathbb{R}$, $\mathfrak{m}_\perp^s = \mathbb{Q}$, $\mathfrak{m}_\perp^v = \mathbb{H}^{n-1}$ with respective eigenvalues 0, $4/\chi$, $1/\chi$, where \mathfrak{m}_\parallel is the centralizer space of e_x in \mathfrak{m} , and $\mathfrak{m}_\perp = \mathfrak{m}_\perp^s \oplus \mathfrak{m}_\perp^v$ is the perp space of \mathfrak{m}_\parallel . Since the linear coframe e provides an identification between \mathfrak{m} and $T_x M$, there is a corresponding decomposition of the tangent spaces $T_\gamma M$ along γ given by

$$\begin{aligned} T_\gamma M &= (T_\gamma M)_\parallel \oplus (T_\gamma M)_\perp, \quad (T_\gamma M)_\parallel = \text{span}(\gamma_x), \\ (T_\gamma M)_\perp &= \text{span}(\gamma_x)^\perp = (T_\gamma M)_\perp^s \oplus (T_\gamma M)_\perp^v \end{aligned}$$

where $(T_\gamma M)_\parallel \simeq \mathbb{R}$, $(T_\gamma M)_\perp^s \simeq \mathbb{Q}$, $(T_\gamma M)_\perp^v \simeq \mathbb{H}^{n-1}$ are the eigenspaces of \mathcal{X}_γ with eigenvalues 0, $4/\chi$, $1/\chi$.

Hereafter, we will write the tangent vector of γ as $T = \gamma_x \in (T_\gamma M)_\parallel$ and the principal normal vector along γ as $N = \nabla_x \gamma_x \in (T_\gamma M)_\perp$; N^s and N^v will denote the projections of N into $(T_\gamma M)_\perp^s$ and $(T_\gamma M)_\perp^v$ respectively.

6.1. mKdV Curve Flow. From the flow (5.9), after undoing the scaling (4.20), we have $\sqrt{\chi}h_\perp = \frac{1}{2}u_x$, $\sqrt{\chi}h_\perp = -u_x$ through (4.9), and $h_\parallel = -\frac{1}{2}u^2 + \frac{1}{2}|\mathbf{u}|^2$ through (4.17). The frame variables (4.1) and (4.2) thereby yield

$$(e_t)_\perp^s = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & \frac{1}{2}u_x & \mathbf{0} \\ \frac{1}{2}u_x & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (e_t)_\perp^v = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & 0 & -\mathbf{u}_x \\ 0 & 0 & \mathbf{0} \\ \bar{\mathbf{u}}_x^t & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (6.11)$$

$$(e_t)_\parallel = (-\frac{1}{2}u^2 + \frac{1}{2}|\mathbf{u}|^2)e_x. \quad (6.12)$$

These quaternionic matrices can be expressed in terms of the geometrical vectors T , N , $\nabla_x N =: N'$ as follows. We first note from (3.17) and (3.18):

$$(e_N)_\perp^s = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & 2u & \mathbf{0} \\ 2u & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} = e \rfloor N^s, \quad (e_N)_\perp^v = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & 0 & -\mathbf{u} \\ 0 & 0 & \mathbf{0} \\ \bar{\mathbf{u}}^t & \mathbf{0} & \mathbf{0} \end{pmatrix} = e \rfloor N^v, \quad (e_N)_\parallel = 0, \quad (6.13)$$

and

$$(e_{N'})_\parallel = (4u^2 - |\mathbf{u}|^2)e_x, \quad (e_{N'})_\perp^s = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & 2u_x & \mathbf{0} \\ 2u_x & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} = e \rfloor (\nabla_x N)_\perp^s, \quad (6.14)$$

$$(e_{N'})_\perp^v = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 & 0 & -\mathbf{u}_x - 3u\mathbf{u} \\ 0 & 0 & \mathbf{0} \\ \bar{\mathbf{u}}_x^t - 3\bar{\mathbf{u}}^t u & \mathbf{0} & \mathbf{0} \end{pmatrix} = e \rfloor (\nabla_x N)_\perp^v. \quad (6.15)$$

In addition, we note:

$$(\text{ad}(e_N)^2 e_x)_\perp = \frac{1}{\sqrt{\chi}^3} \begin{pmatrix} 0 & 0 & -6u\mathbf{u} \\ 0 & 0 & \mathbf{0} \\ -6\bar{\mathbf{u}}^t u & \mathbf{0} & \mathbf{0} \end{pmatrix} = e \rfloor (\text{ad}_x^2(N)T)_\perp^v. \quad (6.16)$$

Since $|\mathbf{u}|^2 = g(N^v, N^v) = |N^v|^2$ and $-4u^2 = g(N^s, N^s) = |N^s|^2$, then from (6.12) and (3.19) we find

$$(e_t)_\parallel = (\frac{1}{8}|N^s|^2 + \frac{1}{2}|N^v|^2)e_x = e \rfloor ((\frac{1}{8}|N^s|^2 + \frac{1}{2}|N^v|^2)T). \quad (6.17)$$

Next, comparing (6.11) to (6.14), (6.15), (6.16), we obtain

$$(e_t)_\perp^s = \frac{1}{4}(e_{N'})_\perp^s = e \rfloor (\frac{1}{4}\nabla_x N)_\perp^s \quad (6.18)$$

and

$$(e_t)_\perp^v = (e_{N'})_\perp^v - \frac{1}{2}\chi(\text{ad}(e_N)^2 e_x)_\perp^v = e \rfloor ((\nabla_x N)_\perp^v - \frac{1}{2}\chi(\text{ad}_x^2(N)T)_\perp^v). \quad (6.19)$$

Thus, from (6.17), (6.18), (6.19) combined with $e_t = e \rfloor \gamma_t$, we derive the following geometrical evolution equation

$$\gamma_t = \frac{1}{4}N'^s_\perp + N'^v_\perp - \frac{1}{2}\chi(\text{ad}_x^2(N)T)_\perp^v + (\frac{1}{8}|N^s|^2 + \frac{1}{2}|N^v|^2)T \quad (6.20)$$

in terms of $T = \gamma_x$, $N = \nabla_x T$, $N' = \nabla_x N$. We can write this evolution equation in an equivalent form without the s and v projections, by relating (6.11) to $\text{ad}(e_x)^{-2}$ applied to (6.13), (6.14), (6.15). This leads to the geometrical evolution equation

$$\chi \gamma_t = \nabla_x (\mathcal{X}_\gamma^{-1} \nabla_x \gamma_x) - (\text{ad}_x^2 (\mathcal{X}_\gamma^{-1} \nabla_x \gamma_x) \gamma_x)_\perp - 3 (\text{ad}_x^2 (\mathcal{X}_\gamma^{-1} \nabla_x \gamma_x) \gamma_x)_\parallel, \quad |\gamma_x| = 1, \quad (6.21)$$

or equivalently

$$\gamma_t = \mathcal{X}_\gamma^{-1} (\chi^{-1} \nabla_x^2 \gamma_x - \frac{1}{2} \text{ad}_x^2 (\nabla_x \gamma_x) \gamma_x)_\perp - \frac{1}{2} \chi^{-1} g(\mathcal{X}_\gamma^{-1} \nabla_x \gamma_x, \nabla_x \gamma_x) \gamma_x, \quad |\gamma_x| = 1, \quad (6.22)$$

called the *non-stretching mKdV map* on $M = \mathbb{H}\mathbb{P}^n$. Each equation (6.20), (6.21), (6.22) is invariant under the isometry group $\text{U}(n+1, \mathbb{H})$ of $M = \mathbb{H}\mathbb{P}^n$.

6.2. SG Curve Flow. The flow given by (5.14) has $w^\perp = \mathbf{w}^\perp = 0$ which implies $w^\parallel = \mathbf{W}^\parallel = 0$ from (4.5). Hence the frame variables (4.3) and (4.4) yield

$$(\omega_t)_\perp = (\omega_t)_\parallel = 0, \quad (6.23)$$

so thus the connection matrix in the flow direction vanishes. This can be expressed geometrically in terms of the tangent vector $T = \gamma_x$ as follows. By applying $\text{ad}(e_x)$ to $\omega_t = 0$, we get

$$0 = \text{ad}(e_x) \omega_t = -[\omega_t, e_x] = e](\nabla_t T) \quad (6.24)$$

through using $D_t e_x = 0$. As a result, we obtain the geometrical evolution equation

$$0 = \nabla_t T \quad (6.25)$$

or equivalently

$$0 = \nabla_t \gamma_x = \nabla_x \gamma_t, \quad |\gamma_x| = 1, \quad (6.26)$$

which is called the *non-stretching wave map* on $M = \mathbb{H}\mathbb{P}^n$. In addition to satisfying the non-stretching property $\nabla_t |\gamma_x| = 0$, this equation (6.26) possesses the conservation law $\nabla_x |\gamma_t| = 0$, corresponding to (5.18). Thus, up to a conformal scaling of t , the evolution given by (6.26) describes a flow with unit speed, $|\gamma_t| = 1$.

The wave map equation (6.26) and its conservation laws are invariant under the isometry group $\text{U}(n+1, \mathbb{H})$ of $M = \mathbb{H}\mathbb{P}^n$.

6.3. Reductions and Curve Flows in $\mathbb{H}\mathbb{P}^1$. We remark that the mKdV and SG curve flows given by the geometric map equations (6.22) and (6.26) have a consistent reduction such that $\gamma(t, x)$ is a map into a submanifold $\mathbb{H}\mathbb{P}^1 \subset \mathbb{H}\mathbb{P}^n$ or into $\mathbb{H}\mathbb{P}^1$ itself when $n = 1$. In the resulting curve flows, the components of the principal normal vector $N = \nabla_x \gamma_x$ along the curve γ in a $\text{U}(1, \mathbb{H}) \times \text{U}(n-1, \mathbb{H})$ -parallel framing satisfy the scalar non-commutative mKdV and SG equations (5.23) and (5.24) respectively.

7. CONCLUDING REMARKS

Our derivation of quaternionic soliton equations (5.6) and their Hamiltonian structure (5.7)–(5.8) from geometric curve flows (6.8) in $\mathbb{H}\mathbb{P}^n$ can be reformulated entirely as an algebraic method at the level of the symmetric Lie algebra structure (2.14)–(2.15) associated with the compact real symplectic group $\text{Sp}(n)$ and the quaternionic unitary group $\text{U}(n, \mathbb{H})$. Specifically, as shown in [22], the Cartan structure equations (3.7)–(3.10) for a framed curve flow in any symmetric space $M = G/H$ arise directly from the zero-curvature equation satisfied by the left-invariant \mathfrak{g} -valued Maurer-Cartan 1-form ω_G on the Lie group G viewed as a principal H -bundle over the manifold M . In this setting the pullback of ω_G by a local

section $\psi : M \rightarrow G$ yields the \mathfrak{m} -valued linear coframe e and \mathfrak{h} -valued linear connection 1-form ω on M via $e + \omega = \psi^*(\omega_G)$, where a change in $\psi \rightarrow \tilde{\psi} = \psi h$ corresponds to a local gauge transformation (3.5) on e and ω .

Such a zero-curvature approach has been used in recent work [32] to derive a quaternionic mKdV equation for the variable $u = \omega_x = \omega|_{\gamma_x}$, based on the choice of a quaternionic connection matrix given by

$$\omega_x = \begin{pmatrix} 0 & 0 & \mathbf{0} \\ 0 & -\mathbf{u} & \mathbf{u} \\ 0 & -\overline{\mathbf{u}}^t & 0 \end{pmatrix}$$

in terms of the imaginary scalar quaternion \mathbf{u} and the $n - 1$ -component vector quaternion \mathbf{u} . This differs compared to our choice given by a parallel framing (3.12) and leads to a more complicated bi-Hamiltonian structure [32]

$$u_t = \tilde{\mathcal{H}}(\varpi^\perp) + \tilde{\mathcal{N}}(h^\perp), \quad \varpi^\perp = \tilde{\mathcal{J}}(h^\perp)$$

where $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{J}}$ are a compatible pair of Hamiltonian cosymplectic and symplectic operators and $\tilde{\mathcal{N}}$ is a non-trivial Nijenhuis operator [29, 31], producing a different form than (5.10)–(5.11) for the quaternionic scalar-vector mKdV equation arising from the flow defined by $h^\perp = u_x$. However, it was not shown in [32] whether the above choice for the connection matrix can be achieved by a gauge transformation starting from an arbitrary form of $u = \omega_x \in \mathfrak{h}$.

Building on work by one of us [33], we will show in a forthcoming paper [34] that these bi-Hamiltonian operators are in fact related to our operators (4.22) and (4.23) by a Backlund transformation that can be interpreted as a Hasimoto gauge transformation corresponding to a change of framing. We will also derive Lax pairs for these quaternionic mKdV equations, as well as the quaternionic SG equations (5.21)–(5.22), by means of the zero-curvature equations (3.7)–(3.10).

In addition, by expanding the imaginary scalar quaternion \mathbf{u} and the $(n - 1)$ component vector quaternion \mathbf{u} in a quaternionic basis $\{1, i, j, k\}$, we plan to compare the resulting coupled integrable systems of $4n - 1$ ordinary (real-valued) scalar variables to other constructions of integrable multi-component mKdV and KdV type systems known in the literature.

Finally, in another direction, we also plan to explore the possibility of deriving octonion soliton equations from bi-Hamiltonian geometric curve flows in the octonion projective plane \mathbb{OP}^2 by adapting the zero-curvature/moving frame method used in the present paper.

REFERENCES

- [1] R. Hasimoto, J. Fluid Mechanics 51, 477–485 (1972).
- [2] G.L. Lamb Jr., J. Math. Phys. 18, 1654–1661 (1977).
- [3] R.E. Goldstein and D.M. Petrich, Phys. Rev. Lett. 67, 3203–3206 (1991).
- [4] A. Doliwa, P.M. Santini, Phys. Lett. A 185, 373–384 (1994).
- [5] U. Pinkall, Result. Math. 27, 328–332 (1995).
- [6] K.-S. Chou and C. Qu, Physica D 162, 9–33 (2002).
- [7] K.-S. Chou and C. Qu, J. Nonlinear Sci. 13, 487–517 (2003).
- [8] G. Marí Beffa, J. Sanders, J.-P. Wang, J. Nonlinear Sci. 12, 143–167 (2002).
- [9] G. Marí Beffa, SIGMA 4, 034 (2008).
- [10] G. Marí Beffa, Physica D 238, 100–115 (2009).
- [11] V.V. Sokolov and T. Wolf, J. Phys. A: Math. and Gen. 34, 11139–11148 (2001).

- [12] S.C. Anco, SIGMA 2, 044, 18 pages (2006).
- [13] S.C. Anco, J. Phys. A: Math. Gen. 39, 2043–2072 (2006).
- [14] S. Anco, T. Wolf, J. Nonlinear Math. Phys. 12, 13–31 (2005); *ibid.* J. Nonlinear Math. Phys. 12, 607–608 (2005).
- [15] J. Langer and R. Perline, Phys. Lett. A 239, 36–40 (1998).
- [16] J. Langer and R. Perline, Pac. J. Math. 195, 157–178 (2000) .
- [17] J. Sanders and J.-P. Wang, Moscow Mathematical Journal 3, 1369–1393 (2003).
- [18] W. Wo and C. Qu, J. Geom. Phys. 57, 1733– (2007).
- [19] G. Marí Beffa, preprint (2009).
- [20] R.L. Bishop, Amer. Math. Monthly 82 (1975) 246–251.
- [21] S.C. Anco, in *Symmetries and overdetermined systems of partial differential equations*, 223–250, IMA Volumes in Math. and Appl. 144 (Springer, New York, 2008).
- [22] S.C. Anco, J. Geom. Phys. 58, 1–37 (2008).
- [23] C. Chevalley, *Theory of Lie Groups* (Princeton U. Press, 1946).
- [24] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces* (Amer. Math. Soc., Providence, 2001).
- [25] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry* Vol. I and II, (Wiley 1969).
- [26] R.W. Sharpe. *Differential geometry*, Vol. 166 Graduate Texts in Mathematics (Springer-Verlag, New York, 1997).
- [27] H. Guggenheimer, *Differential Geometry* (McGraw Hill, 1963).
- [28] P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York 1986).
- [29] I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations* (Wiley 1993).
- [30] F. Magri, J. Math. Phys. 19, 1156–1162 (1978).
- [31] F. Magri, Lecture Notes in Physics Vol. 120 (Springer, Berlin-New York, 1980).
- [32] E. Asadi and J. Sanders, in *SPT 2007—Symmetry and perturbation theory*, 21–28 (World Sci., Hackensack, NJ, 2008).
- [33] E. Asadi, PhD Thesis, Vrije Universiteit, Amsterdam 2008.
- [34] S.C. Anco and E. Asadi, In preparation (2009).

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